Harish-Chandra Center

- $sl_2$ has no center
- but $U(sl_2)$ does, $\mathcal{O}[C]$, $c = 2xy + 2yx + H^2$

Ex: Check that $c$ is central.

- $c$ is a "quantisation" of the Killing form

Explanation:

$U(sl_2)$ has natural filtration $U'/U''$ where $U''$ is nilpotent and $U'$ is nilpotent and $U''$ is nilpotent and $U''$ is nilpotent and $U''$ is nilpotent.

and $\text{Gr}(U(sl_2)) \cong U''/U'' = \mathcal{O}[sl_2]$.

The Lie bracket induces a Poisson bracket on $\mathcal{O}[sl_2]$, so $U(sl_2)$ is the "quantisation" of $\mathcal{O}[sl_2]$ as a Poisson alg.

and $c \in \text{Gr}(U(sl_2))$ is a Killing Form (up to scale, maybe)

Thus picture exhibits the Poisson alg. over its center w/ symplectic leaves.

$$Z = \mathcal{O}[C] \subset U(sl_2)$$

Introduce $\mathcal{B} = \mathbb{C}^n \times \mathbb{C}^n \subset sl_2$

$A = \mathbb{C}^n \subset U(sl_2)$

Consider $\text{N} = \text{module } U(sl_2) \otimes_{\mathbb{C}} \mathbb{C}$ (Principal Serre)
**Proof of Theorem that Rep.(S^2) is s.s.**

Claim: any SES \( 0 \to U \to V \to L_0 \to 0 \) splits if \( U \) is irrep.

Proof:

i) \( U \cong L_0 \) Exercise

ii) \( L_0 \neq 0 \) - use action of resvar to split

Note: (an extra claim for \( U \cong \) irmps (Exercise)

Now suppose general SES \( 0 \to U \to W \to V \to 0 \)

Consider \( \text{Hom}_g(V, U) \sim \): subspace that scale \( U \)

Get SES: (in Rep(S^2)) \( 0 \to Z \to Y \to 0 \)

(Extended) Claim \( Y \cong Z \oplus U \) \( \implies \) \( \exists \) non zero scaling split

**Remark Recall RCG**

\[ 0 \to U \cong V \to L_0 \to 0 \]

**This doesn't split:** (Claim in) \( \text{foils} \) in \( \infty \)-dim.
What is the Alg. Geom. of all $U_{\mathfrak{sl}_2}$-mod

$\mathfrak{sl}_2 \rightarrow \text{Vect} (P')$

Local formulas

$x \rightarrow -z^2 \partial_z$
$H \rightarrow 2z \partial_z$
$\gamma \rightarrow \partial_z$

Other patch

$x \rightarrow 2w$
$H \rightarrow -2w \partial_w$
$\gamma \rightarrow -w^2 \partial_w$

Def: Diff op on $X = \text{Spec} R$ is an element

$D_x = \bigcup_{n=0} \text{Diff}^n \subset \text{End}_R (R)$

Check $D_x^{\geq 0} = R$

Ex: $D_0 (D_x) = \mathcal{O}(T^* X)$

First fundamental fact: $\mathfrak{sl}_2 \rightarrow \text{Vect} (P')$ induces an iso between $\mathfrak{sl}_2 / \mathfrak{g}_0 \rightarrow D_{P'}$ (nonproper $\Gamma (P', D_{P'})$)

when $\mathfrak{g}_0 = \text{ideal of } \mathfrak{g} \text{ coming from trivial rep}$

(why, that kill trivial $\Gamma$)

Ex: $D_x$ is “enveloping algebra” of $\mathfrak{g}_0 T_x$
$U(sl_2/\mathbb{Z}) \to \text{func}_{\mathfrak{g}} \quad \mathfrak{g} = \text{det}^r(0)$

$\mathfrak{g}$ is a resolution of $\mathfrak{g}$

$T^* \mathbb{P}^r \to \mathcal{N}$

The fundamental fact is the correspondance

"quantum" statement