Harish-Chandra Center

$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- sl_2 has no center
  - but Usl_2 does, $\mathbb{C}[c]$, $c = 2XY + 2YX + H^2$

<u>Ex</u>: Check that c is central.

- c is a "quantisation" of the Killing form

<u>Explanation:</u>

Usl_2 has natural filtration $\bigcup_{n\geq 1} U_n$ $\quad U_n =$ ~~monomials~~ elements expressible in monomials of $\leq n$ degree

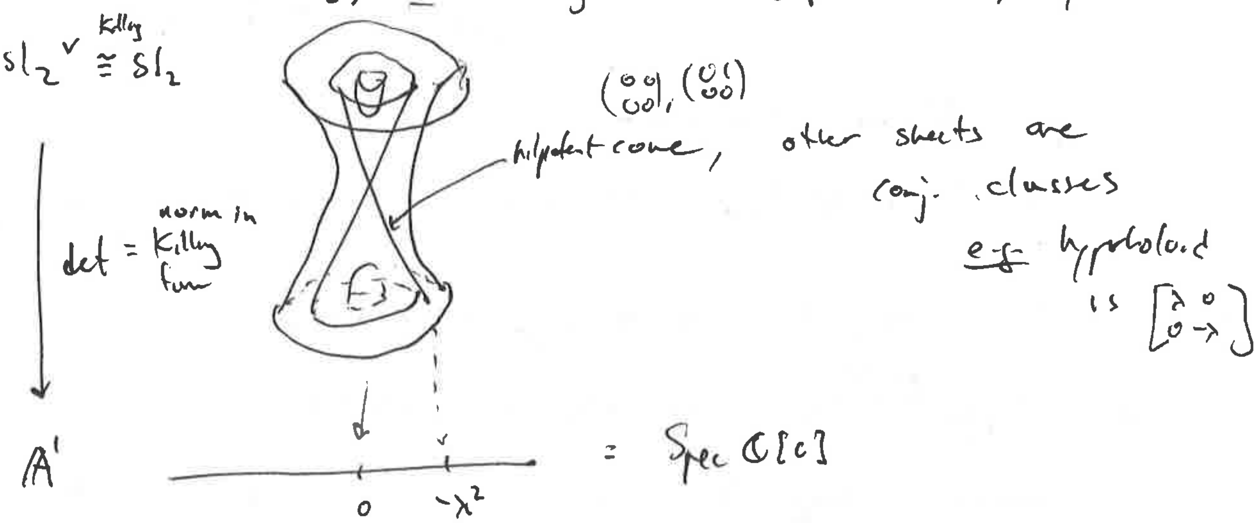
and $Gr(Usl_2) \bigoplus_{n\geq 0} U_n/U_{n-1} = \mathbb{C}[sl_2^\vee] = Sym(sl_2)$

the Lie bracket induces a Poisson Bracket on $\mathcal{J}$

so Usl_2 is the "quantisation" of $\mathbb{C}[sl_2^\vee]$ as a Poisson alg.

and $c \in Gr(Usl_2)$ <u>is</u> Killing Form (up to scale, maybe)

$sl_2^\vee \overset{\text{Killing}}{\cong} sl_2$



This picture exhibits the Poisson alg. over its center w/ symplectic leaves.

$\mathfrak{Z} = \mathbb{C}[c] \subset Usl_2$

Introduce $\mathfrak{b} = \langle H, X \rangle \subset sl_2$

$\mathfrak{n} = \langle X \rangle \subset sl_2$

$\mathfrak{h} = \mathfrak{b}/\mathfrak{n}$ $\quad \mathbb{C}[H] = U\mathfrak{h}$

Consider Usl_2-module $\quad Usl_2 \otimes_{U\mathfrak{n}} \mathbb{C}$ $\quad$ (Principal Series)

~~Recall: General fact. Any endomorphism of~~

Ex: Check that the left action of $\mathbb{C}[c]$ is given by the right action of $\mathbb{C}[(H+1)^2-1]$

$$\mathbb{C}[(H+1)^2-1] = \mathbb{C}[H+1]^{\mathbb{Z}_2}$$

$\mathbb{Z}_2$ ← flip across -1

Spec $\mathbb{C}[H]$

Conclusion: $\operatorname{Spec} \mathfrak{Z} = \operatorname{Spec} \mathbb{C}[H] / \mathbb{Z}_2$ ← reflection at -1

Ex: c scales L_n, $n \geq 0$
by $(n+1)^2-1$

Proof of Theorem that $\operatorname{Rep}_{f.d.}(\mathfrak{sl}_2)$ is s.s.

Claim: any SES $0 \to U \to V \to L_0 \to 0$ splits if U is irrep

Proof: i) $U = L_0$ Exercise

ii) U = irrep $\neq L_0$ – use action of casimir to split □

Note: Can extend claim for $U = \bigoplus$ irreps (Exercise)

Now suppose general SES $0 \to U \to W \to V \to 0$

Consider $\operatorname{Hom}_k(V, U) \supset Y :=$ subspace that scale U
$\cup$
$Z :=$ subspace that kill U

Get SES: (in $\operatorname{Rep}_{fd}(\mathfrak{sl}_2)$)

$$0 \to Z \to Y \to \mathbb{C} \to 0$$

(Extended) Claim $\Rightarrow$ $Y \cong Z \oplus \mathbb{C}$ $\Rightarrow$ $\exists$ non zero scaling splitting □

Remark: Recall BGG

$$0 \to V_{-1} \to V_0 \to L_0 \to 0$$

$V_{-1} = Y\mathbb{C}\langle Y \rangle$, $V_0 = \mathbb{C}\langle Y \rangle$

This doesn't split: (Claim ii) fails) in ∞-dim

What is the Alg. Geom of all Usl_2-mod

BWB ⇝ fin-dim

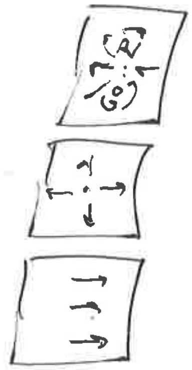
Beil. Bern ⇝ all reps.

$$\mathbb{P}^1 = \mathbb{A}^1_z \sqcup_{\mathbb{G}_m} \mathbb{A}^1_w \qquad sl_2 \longrightarrow Vect(\mathbb{P}^1)$$

$w = \frac{1}{z}$

Local formulas

$$X \longmapsto -z^2 \partial_z$$
$$H \longmapsto 2z\partial_z$$
$$Y \longmapsto \partial_z$$



Other patch

$$X \mapsto \partial_w$$
$$H \mapsto -2w\partial_w$$
$$Y \mapsto -w^2\partial_w$$

Def Diff op on $X = \operatorname{Spec} R$ is an element

of $D_X = \bigcup_{n \geq 0} D_X^{\leq n}$ $\quad D_X^{\leq n} \subset End_k(R)$ st.

$$[D_X^{\leq n}, R] \subset D_X^{\leq n-1}$$

Check $D_X^{\leq 0} = R$

Ex: $Gr(D_X) = \mathcal{O}(T^*X)$

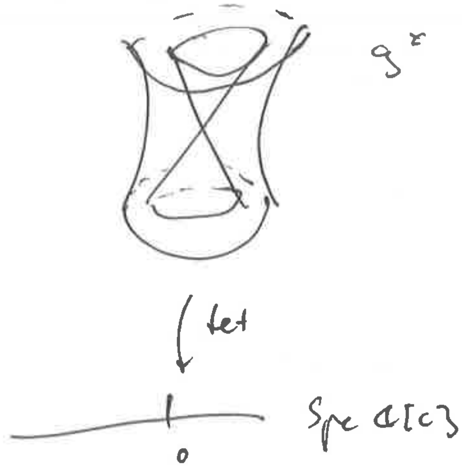
First fundamental fact: $sl_2 \longrightarrow Vect(\mathbb{P}^1)$ induces an iso

between $Usl_2/\mathfrak{Z}_0 \xrightarrow{\sim} D_{\mathbb{P}^1}$ (more properly $\Gamma(\mathbb{P}^1, D_{\mathbb{P}^1})$)

where $\mathfrak{Z}_0$ = ideal of $\mathfrak{Z}$ corresponding to trivial rep (things that kill trivial rep)

Ex: D_X is "enveloping algebroid" of T_X

Picture



$U\mathfrak{sl}_2/\mathfrak{z}_0 \rightsquigarrow$ functions on  $= \det^{-1}(0)$

$\exists$ resolution of $\mathcal{N}$

$$T^*\mathbb{P}^1 \longrightarrow \mathcal{N}$$

The fundamental fact is the corresponding "quantised" statement