Harish-Chandra center

The remarkable fact is that $sl(2)$ has no center, but its enveloping algebra has a center, a polynomial algebra $\mathbf{C}[c]$, where $c = 2XY + 2YX + H^2$.

Exercise 1. Show that c is central.

If you had a nilpotent algebra, we would not have a center. I want to explain that c is a "quantization" of the Killing form. The enveloping algebra $\mathrm{U}sl(2)$ has a natural filtration by the sets U_n of elements expressable in monomials of at most n variables. The algebra structure is compatible with this filtration. We can compute its associated graded.

$$\mathrm{Gr}(\mathrm{U}sl(2)) = \bigoplus_n \mathrm{U}_n/\mathrm{U}_{n-1}$$

You can do this by hand or use the Poincaré-Birkhoff-Witt theorem and the answer is that this is

$$\mathrm{Sym}\, sl(2) = \mathcal{O}(sl(2)^*)$$

The memory of the associative algebra you see on the associated graded is a Poisson bracket, so the original Lie bracket induces a Poisson bracket on $\mathcal{O}(sl(2)^*)$. $\mathrm{U}g$ is the "quantization" of $\mathcal{O}(sl(2)^*)$ as a Poisson algebra. For linear things the Poisson bracket is the Lie algebra and then you extend it by the Leibniz rule. The Poisson structure is on the associated graded. [Picture: Poisson algebra over 0, associated algebra over $\hbar \neq 0$]

(See the other notes for this lecture for some of the pictures.) Now I will tell you what I mean by the element c being a quantization. The Casimir $c \in \mathrm{Gr}_2(\mathrm{U}sl(2))$ is the Killing form up to possible scale.

I would like to draw a picture to explain how you might have guessed that this is the center. Let me draw a picture of $sl(2)^*$. I have drawn a cone, a quadratic cone, and now I'll draw some non-singular quadratic hypersurfaces. $sl(2)^*$ has a natural function on it called the Killing form, it is a map to $\mathbb{A}^1$ and the determinant is the Killing form as a quadratic form. The map $sl(2)^* \to \mathbb{A}^1$ is a map from a Poisson algebra to its Poisson center. The fibers are symplectic leaves. Find the Poisson center and then quantize it. The Killing form was in the Poisson center already before we quantized.

Let's go back to being more concrete. I want to give a way to think the Harish-Chandra theorem.

$$\mathfrak{z} = \mathbf{C}[c] \subseteq Usl(2)$$

Let's introduce the following notation for upper triangular matrices and strictly upper triangular matrices.

$$\mathfrak{b} = \langle H, X \rangle \subset sl(2)$$
$$\mathfrak{n} = \langle X \rangle \subset sl(2)$$
$$\mathrm{U}\mathfrak{h} = \mathbf{C}[H]$$

Consider the $\mathrm{U}sl(2)$-module $\mathrm{U}sl(2) \otimes_{\mathrm{U}\mathfrak{n}} \mathbf{C} = \mathrm{U}sl(2)/\mathrm{U}\mathfrak{n}$. Note that this is a $\mathrm{U}sl(2)$ left module, it is also a $U\mathfrak{h}$ right module.

Exercise 2. Check that the left action of the center $\mathbf{C}[c]$ is given by the right action of $\mathbf{C}[(H+1)^2]$.

Let's do the exercise. Let's act in this principal series module on the element 1. If an X is all the way to right, it's zero.

$$c \cdot 1 = 2XY + H^2 = 2(YX + H) + H^2 = 2H + H^2 = (H+1)^2 - 1$$

$$\mathbf{C}[(H+1)^2 - 1] = \mathbf{C}[(H+1)^2] \qquad \mathbf{C}[H+1]^{\mathbf{Z}/2} = \mathbf{C}[(H+1)^2]$$

[picture of Spec $\mathbf{C}[H]$ with integer points labeled.]

Conclusion:

$$\operatorname{Spec} \mathfrak{z} = \operatorname{Spec}(\mathbf{C}[H+1])/(\mathbf{Z}/2)$$

I'll use the existence of the Harish-Chandra center to prove that finite-dimensional $sl(2)$-representations are semi-simple.

Exercise 3. c scales $L_n, n \geq 0$ by $2n + n^2 = (n+1)^2 - 1$

[Picture of this parabola]

What this picture tells you that the representations are far away from each other.

[another picture, L_0 is at 0, L_1 is at 3]

I am going to now give a proof of the theorem that $Rep_{fd}(sl(2))$ is semi-simple. I told you earlier that this equivalent to the statement that the radical is trivial, but I am not going to prove that.

Claim: Any SES

$$0 \to U \to V \to L_0 \to 0$$

with U irreducible splits.

(i) $U = L_0$: Any two-dimensional representation of $sl(2)$ with 1-dim. sub is trivial.

(ii) U irreducible, $\neq L_0$: The Casimir acts with different scalars on the sub and the quotient.

Exercise 4. (i) and (ii) imply the claim in general.

Now suppose we have a general SES

$$0 \to U \to W \to V \to 0$$

Consider $\operatorname{Hom}_{\mathbf{C}}(V, U)$ and define the subspace Y of maps that scale U. Inside of Y, look at the subspace Z of maps that kill U. We have a SES

$$0 \to Z \to Y \to \mathbf{C} \to 0$$

of $sl(2)$-reps. The Claim implies that $Y \simeq Z \oplus \mathbf{C}$. This completes the proof as we now have a map from V to U in the complement of the subspace of maps filling U.

Remark. Recall the BGG resolution

$$0 \to V_{-1} \to V_0 \to L_0 \to 0$$

where $V_0 = C\langle Y \rangle$ and $V_{-2} = YC\langle Y \rangle$. This doesn't split. The Casimir acts by the same scalar on V_{-2} and L_0. So (ii) in the proof of the Claim fails in the infinite-dimensional case.

I think we have completed an arc about *finite-dimenionsal* representations of $sl(2)$ [...] What is the algebraic geometry of all $\mathrm{U}sl(2)$-modules?

Analogy: Fin dim : All mods :: Borel-Weil-Bott : Beilinson-Bernstein

We are going to construct algebraic-geometric objects on $\mathbf{P}_1$.

$$\mathbf{P}_1 = \mathbf{A}^1_z \sqcup_{\mathbf{G}_m} \mathbf{A}^1_w \qquad sl(2) \to \mathrm{Vect}(\mathbf{P}^1)$$

Local formulas

$$\begin{aligned} X &\mapsto -z^2\partial_z \\ H &\mapsto 2z\partial_z \\ Y &\mapsto \partial_z \end{aligned}$$

Y is the translation vector field, H is 2 times the dilation [Pictures of these vector fields].

In the other coordinate patch $w = 1/z$,

$$\begin{aligned} X &\mapsto \partial_w \\ H &\mapsto -2w\partial_w \\ Y &\mapsto -w^2\partial_w \end{aligned}$$

What is the enveloping algebra?

Definition. A differential operator on $X = \operatorname{Spec} R$ is an element of

$$D_X = \bigcup_{n \geq 0} D_X^{\leq n}$$

where $D_X^{\leq n} \subset \mathrm{End}_k(R)$ s. t.

$$D_X^{\leq -1} = \{0\}$$

and

$$[D_X^{\leq n}, R] \subseteq D_X^{\leq n-1}$$

Note: $D_X^{\leq 0} = R$

Exercise 5. $\mathrm{Gr}(D_X) = \mathcal{O}(T^*X)$

Exercise 6. D_X is the enveloping algebroid of T_X.

First fundamental fact: $sl(2) = \mathrm{Vect}(\mathbf{P}^1)$ induces

$$\mathrm{U}sl(2)/\mathfrak{z}_0 \cong \Gamma(\mathbf{P}^1, D_{\mathbf{P}^1})$$

$\mathfrak{z}_0 =$ ideal of $\mathfrak{z}$ for trivial rep L_0 (things that kill the trivial representation)

[Another picture of g^*, and the map down to Spec $\mathbf{C}[c]$. What is the classical interpretation of $\mathrm{U}sl(2)/\mathfrak{z}_0$? nilpotent cone]

There is a natural resolution of the nilpotent cone, the cotangent bundle of $T^*\mathbf{P}^1$. The l. h. s. is a quantization of $\mathcal{O}(N)$, functions on the nilpotent cone, and the r. h. s. is a quantization of $\tilde{N} = T^*\mathbf{P}^1$.