No class next Thursday 2/2

Today: Rep Theory of $sl(2)$

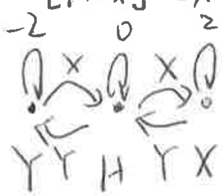
Warm up: $sl(2) = so(3)$

Construct the map $sl(2) \to so(3) \subset gl(V)$ $\dim V = 3$, V has a quadratic form.

Take V = adj rep of $sl(2)$.

Basis of $sl(2)$: $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$[H, X] = 2X$ $[H, Y] = -2Y$ $[X, Y] = H$



action on subspaces.

The quadratic form Q = Killing form $Q(A, B) = B_V(A, B) = Tr_V(AB)$

(Exercise: B_V is an invariant degenerate Symmetric Bilinear form)

(Exercise: $sl(2) \to so(3)$, by it is injective. By dimension, it is iso)

Groups: $SL(2) \to SL(2)/\text{center} = SO(3)$

s.c form adj form.

Describe $Rep_{f.d} \, sl(2) = U sl(2)\text{-mod}_{f.d}$

Recall: $Rep_{f.d} \, sl(2)$ is s.s. (is equivalent to the fact that $rad(sl(2))$ is trivial)

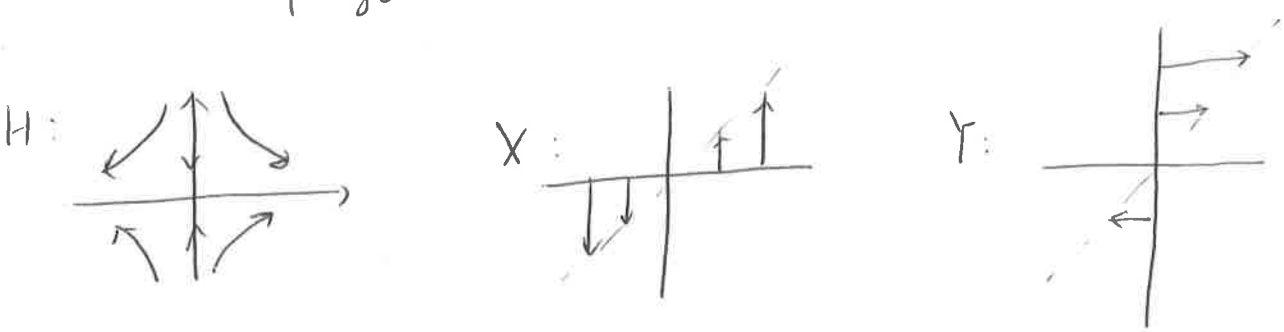
Simple/irreducible reps (Geometrically) Lie algs are vector fields.

$SL(2) \circlearrowright \mathbb{A}^2$ or $SL(2) \to Aut(\mathbb{A}^2) \overset{\text{diff}}{\rightsquigarrow} sl(2) \to Vect(\mathbb{A}^2)$

$\mathbb{A}^2 = Spec\, \mathbb{C}[x, y]$ $H = x\partial x - y\partial y$

$X = x\partial y$

$Y = y\partial x$



So $sl(2)$ acts on $\mathcal{O}(\mathbb{A}^2) = \bigoplus_n \mathcal{O}_n(\mathbb{A}^2) = \bigoplus_n \mathbb{C}\langle x^n, x^{n-1}y, \dots y^n\rangle$

Action is homogenous $\rightarrow$ preserve degree $\& \oplus$

Def: $\bigoplus_n \mathbb{C}\langle x^n, x^{n-1}y, \dots, y^n\rangle = \bigoplus_n L_n$.

Thm: Simple in $Rep_{f.d}\, sl(2)$ are precisely L_n. $L_n \cong L_n^*$

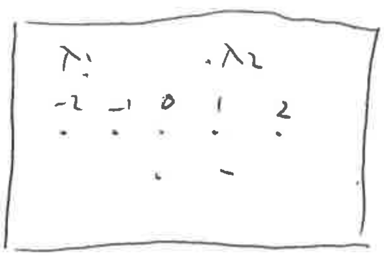
Ex: L_0 = trivial L_1 = Standard L_2 = adjoint

(Ex: Symmetric when n even
skew symmetric when n odd)

$sl(2) \rightarrow so(L_n)$ n even
$sl(2) \rightarrow sp(L_n)$ n odd.

Idea of the Proof: Highest weight analysis

V. f.d. irreducible. Restrict $\mathbb{C}[H] \subset U sl(2)$.



Spec $\mathbb{C}[H]$

Consider integral point.

Choose highest wt $\lambda_{h.w}$, a highest weight vector $v_{h.w} \in V_{\lambda_{h.w}}$.

Generate by applying X, Y to $v_{h.w}$. Note that $Xv_{h.w} = 0$.

$Y \cdot v_{\lambda_{h.w}} \in V_{\lambda_{h.w}-2}$. But it will terminal. ~~So $\lambda_{h.w} = 2k$~~.

Alg geom interpretation (BWB)

$\mathbb{P}^1 = (\mathbb{A}^2 \setminus \{0\}) / \mathbb{G}_m$ flag var

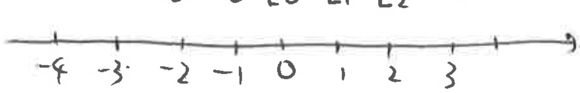
Borel-weil for $sl(2)$.

$n \geq 0$ $\mathcal{O}_n(\mathbb{A}^2) = \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) = L_n$

$n \leq -1$ $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) = L_{-n-2}$

$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(<0)) = 0$ $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\geq -1)) = 0$

H^0 $\cdots$ 0 0 L_0 L_1 L_2 $\cdots$



H^1 $\cdots$ L_2 L_1 L_0 0 0 $\cdots$

Recall calc of H^1 $\quad \mathbb{P}^1 = \mathbb{A}^1 \amalg_{\mathbb{G}_m} \mathbb{A}^1$

Cech complex

$$\begin{array}{ccc} 1 & \nearrow^{1} \ \mathbb{C}[x,y]/(xy=1) & \nwarrow^{y^n ?} \\ 0 \ \ 1 & \mathbb{C}[x] \oplus \mathbb{C}[y] & 1 \end{array}$$

for $H^*(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-n))$.

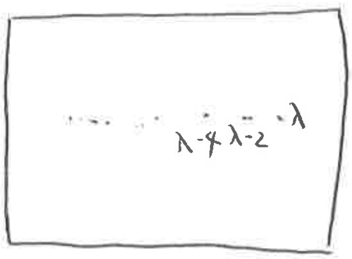
Infinite-dim rep.

Verma module: $\quad V_\lambda = U\mathfrak{sl}(2) \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda$

$\lambda \in \operatorname{Spec} \mathbb{C}[H]$

$\mathfrak{b} = \mathbb{C}\langle H, X\rangle$ upper triang

$H_\lambda \circlearrowright \mathbb{C}_\lambda \quad X \underset{0}{\circlearrowright} \mathbb{C}_\lambda$.



V_λ is a universal h.w. module

$\operatorname{Hom}_{\mathfrak{sl}(2)}(V_\lambda, W) = \operatorname{Hom}_{U\mathfrak{b}}(\mathbb{C}_\lambda, W)$

↑ ~~Sup~~ Subspace where X acts by 0, H acts by scalar

BGG-resolution.

$$0 \leftarrow L_n \leftarrow V_n \leftarrow V_{-n-2} \leftarrow 0$$

Exercise V_λ is irr if $\lambda \notin \mathbb{N}$.

Caution: Not all irr reps are h.w.

Ex: $V = \mathbb{C}\langle \ldots, x^{\lambda-1}y, x^\lambda, x^{\lambda+1}y^{-1}, \ldots \rangle \quad \lambda \notin \mathbb{Z}$

First step in understanding all reps

$\mathfrak{sl}(2)$ has trivial center but $U\mathfrak{sl}(2)$ has center $\mathbb{C}[c]$

↖ Casimir element

$$c = 2XY + 2YX + H^2$$

Exercise: Check c is central.