Langlands Duality Lecture 3

January 24, 2017

Today we will discuss the representation theory of $\mathfrak{sl}(2)$. As usual, we work over $k = \mathbb{C}$. We will be using the basis given by

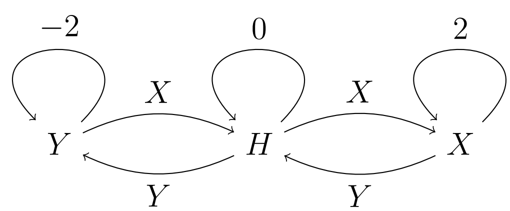
$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

The Lie bracket in this basis is given by

$$\begin{aligned} [H, X] &= 2X \\ [H, Y] &= -2Y \\ [X, Y] &= H \end{aligned}$$

As a warmup, we show that $\mathfrak{sl}(2)$ is isomorphic to $\mathfrak{so}(3)$. To show that, we look for a representation $\mathfrak{sl}(2) \to \mathfrak{gl}(V)$ on a 3-dimensional vector space V with a nondegenerate, symmetric bilinear form Q preserved by $\mathfrak{sl}(2)$.

We may choose V to be the adjoint representation of $\mathfrak{sl}(2)$. Observe that, by the above relations, we may draw the action on subspaces as follows



We then choose Q to be the Killing form $B_V(A, B) = \mathrm{tr}_V(AB)$.

Exercise: B_V is an invariant nondegenerate symmetric bilinear form on V.

Exercise: Conclude that the resulting map $\mathfrak{sl}(2) \to \mathfrak{so}(3)$ is an isomorphism.

At the level of groups, you get

$$SL(2) \to SL(2)/\text{center} = SO(3)$$

Moreover, $SL(2)$ is simply connected with center $\mathbb{Z}/2$, and so $SO(3) = SL(2)/(\mathbb{Z}/2)$ is its adjoint form.

Representations

We next describe the finite dimensional representations of $\mathfrak{sl}(2)$ (equivalently, of $\mathcal{U}\mathfrak{sl}(2)$). Recall that the category $\mathrm{Rep}_{\mathrm{f.d.}}(\mathfrak{sl}(2))$ is semisimple, a property which is equivalent to the radical of $\mathfrak{sl}(2)$ being trivial.

To find the simple/irreducible representations, we think geometrically (think of Lie algebras as algebras of vector fields). Consider the action

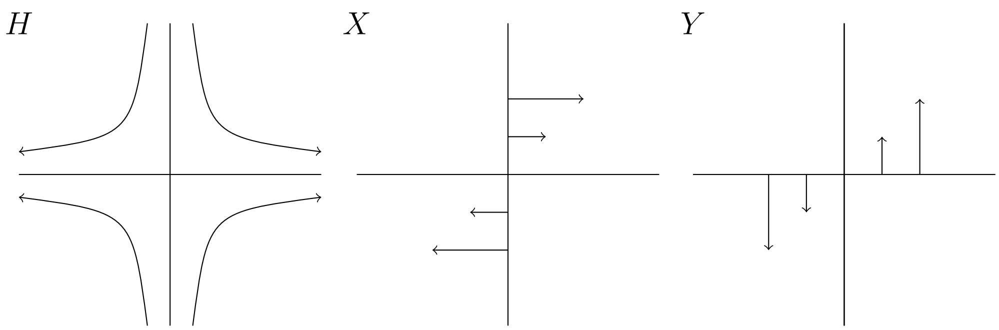
$$SL(2) \to Aut(\mathbb{A}^2)$$

By differentiating this we obtain an action

$$\mathfrak{sl}(2) \to Vect(\mathbb{A}^2)$$

given by

$$\begin{aligned} H &= x\partial_x - y\partial_y \\ X &= y\partial_x \\ Y &= x\partial_y \end{aligned}$$



Therefore $\mathfrak{sl}(2)$ acts on $\mathcal{O}(\mathbb{A}^2)$. Since the action is homogeneous, we get a decomposition

$$\mathcal{O}(\mathbb{A}^2) = \bigoplus_n \mathcal{O}_n(\mathbb{A}^2) = \bigoplus_n \mathbb{C}\langle x^n, x^{n-1}y, \dots, y^n\rangle = \bigoplus_n L_n$$

as $\mathfrak{sl}(2)$-modules.

Theorem. *The simples in* $\mathrm{Rep}_{\mathrm{f.d.}}(\mathfrak{sl}(2))$ *are precisely* L_n. *Moreover, we have* $\dim L_n = n+1$, $L_n \simeq L_n^*$, *and* L_n *is the symmetric algebra of the standard representation.*

Observe that L_0 is the trivial representation, L_1 is the standard representation, and L_2 is the adjoint representation.

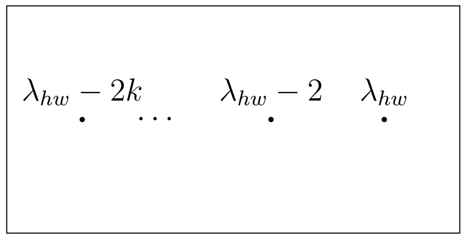
<u>Exercise</u>: The isomorphism $L_n \simeq L_n^*$ (well defined up to a constant) defines a bilinear form which is symmetric for n even and skew-symmetric for n odd.

From $n = 1$ and $n = 2$ this gives us isomorphisms $\mathfrak{sl}(2) = \mathfrak{sp}(1)$ and $\mathfrak{sl}(2) = \mathfrak{so}(3)$. In general, you get maps from $\mathfrak{sl}(2)$ into $\mathfrak{so}(L_n)$ when n is even, and $\mathfrak{sp}(L_n)$ when n is odd.

Idea of the proof of the theorem: Highest weight analysis.

Let V be finite dimensional, irreducible. Restrict to $\mathbb{C}[H] \subset \mathcal{U}\mathfrak{sl}(2)$. Decompose $V = \bigoplus_1^k V_{\lambda_i}$ where λ_i are the eigenvectors of H. We think of the λ_i as belonging to $\operatorname{Spec}\mathbb{C}[H]$. Choose a highest weight λ_{hw} (that is, an eigenvalue with maximal real part), and highest weight eigenvector $v_{hw} \in V_{\lambda_{hw}}$.

We have $Xv_{hw} = 0$ and $Yv_{hw} \in V_{\lambda_{hw}-2}$. One may see that the vectors $v_{hw}, Yv_{hw}, Y^2v_{hw}, \ldots$ span V, and have eigenvalues $\lambda_{hw}, \lambda_{hw}-2, \lambda_{hw}-4, \ldots$. In $\operatorname{Spec}\mathbb{C}[H]$ we have a picture like this



Exercise: Show that V is isomorphic to L_n for some $n \geq 0$.

Algebro-geometric interpretation (Borel-Weil-Bott)

Consider the flag variety $\mathbb{P}^1 = (\mathbb{A}^2\backslash\{0\})/\mathbb{G}_m$. For $n \geq 0$, we have

$$L_n = \mathcal{O}_n(\mathbb{A}^2) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$$

which is what Borel-Weil says for $\mathfrak{sl}(2)$.

For $n < -1$, we also have Bott for $\mathfrak{sl}(2)$, which says

$$H^1(\mathbb{P}^1, \mathcal{O}^1_{\mathbb{P}}(n)) = L_{-n-2}$$

You also have

$$H^0(\mathbb{P}^1, \mathcal{O}^1_{\mathbb{P}}(n)) = 0$$

for negative n, and

$$H^1(\mathbb{P}^1, \mathcal{O}^1_{\mathbb{P}}(n)) = 0$$

for $n \geq -1$. Putting this all together, the cohomology is as follows

H^1 :	L_1	L_0	0	0	0	0	0
H^0 :	0	0	0	L_0	L_1	L_2	L_3
n :	-3	-2	-1	0	1	2	3

(the symmetry here is to be expected by Serre duality).

Recall that to compute H^1 we may use the Cech complex. Write $\mathbb{P}^1 = \mathbb{A}^1 \amalg_{\mathbb{G}_m} \mathbb{A}^1$. Then the following complex computes the cohomology

$$0 \to \mathbb{C}[x] \oplus \mathbb{C}[y] \to \mathbb{C}[x,y]/(xy=1) \to 0$$

where the middle arrow is defined by $(1,0) \mapsto 1$ and $(0,1) \mapsto y^{-n}$.

What about infinite dimensional representations?

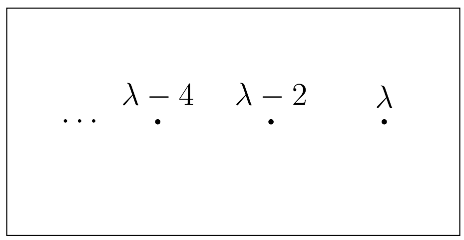
They are very useful, even if you are primarily interested in finite dimensional representations.

Verma Modules

Let $\lambda \in \operatorname{Spec} \mathbb{C}[H]$. Let $B = \mathbb{C}\langle H, X\rangle$ be the subalgebra of $\mathfrak{sl}(2)$ consisting of upper triangular matrices. Let $\mathbb{C}_\lambda$ be $\mathbb{C}$ considered as a B-module, where H acts by multiplication by λ, and X acts by zero. The Verma module V_λ is then defined as

$$V_\lambda = \mathcal{U}\mathfrak{sl}(2) \otimes_{\mathcal{U}B} \mathbb{C}_\lambda$$

This has weights $\lambda, \lambda - 2, \lambda - 4, \ldots$, giving the following picture in $\operatorname{Spec} \mathbb{C}[H]$



Moreover, $1 \otimes 1 \in V_\lambda$ is the universal highest weight vector of weight λ. In other words, for any representation W of $\mathfrak{sl}(2)$ we have

$$\operatorname{Hom}_{\mathfrak{sl}(2)}(V_\lambda, W) = \operatorname{Hom}_{\mathcal{U}B}(\mathbb{C}_\lambda, W)$$

which is in correspondence with vectors w of W of weight λ such that $Xw = 0$.

The Verma modules appear in the BGG (Bernstein, Gelfand, Gelfand) resolution of L_n

$$0 \leftarrow L_n \leftarrow V_n \leftarrow V_{-n-2} \leftarrow 0$$

In particular, V_n is not irreducible when $n \in \mathbb{N}$.

Exercise: Show that V_λ is irreducible if $\lambda \notin \mathbb{N}$.

Caution: Not all irreducible representations are generated by higher weight vectors. For an example, consider

$$V = \mathbb{C}\langle \ldots, x^{\lambda-1}y, x^\lambda, x^{\lambda+1}y^{-1}, \ldots \rangle$$

where $\lambda \notin \mathbb{Z}$.

First step in understanding all representations

The Lie algebra $\mathfrak{sl}(2)$ has trivial center, but its enveloping algebra $\mathcal{U}\mathfrak{sl}(2)$ has center $\mathbb{C}[c]$, where c is the Casimir element

$$c = 2XY + 2YX + H^2$$

This element can be thought of as a quantization of the Killing form.

Exercise: c is central.