Today we will discuss the representation theory of $\mathfrak{sl}(2)$. As usual, we work over $k = \mathbb{C}$. We will be using the basis given by

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

The Lie bracket in this basis is given by

$$[H, X] = 2X$$

$$[H, Y] = -2Y$$

$$[X, Y] = H$$

As a warmup, we show that $\mathfrak{sl}(2)$ is isomorphic to $\mathfrak{so}(3)$. To show that, we look for a representation $\mathfrak{sl}(2) \rightarrow \mathfrak{gl}(V)$ on a 3-dimensional vector space $V$ with a nondegenerate, symmetric bilinear form $Q$ preserved by $\mathfrak{sl}(2)$.

We may choose $V$ to be the adjoint representation of $\mathfrak{sl}(2)$. Observe that, by the above relations, we may draw the action on subspaces as follows

$$\begin{array}{cccc}
-2 & & 0 & \text{X} \\
& \text{Y} & & \\
\text{X} & & \text{H} & \text{X} \\
& \text{Y} & & \\
& & 0 & \text{X} \\
& & \text{X} & \text{Y} \\
2 & & & \\
\end{array}$$

We then choose $Q$ to be the Killing form $B_V(A, B) = \text{tr}_V(AB)$.

**Exercise:** $B_V$ is an invariant nondegenerate symmetric bilinear form on $V$.

**Exercise:** Conclude that the resulting map $\mathfrak{sl}(2) \rightarrow \mathfrak{so}(3)$ is an isomorphism.

At the level of groups, you get

$$SL(2) \rightarrow SL(2)/(Z/2) = SO(3)$$

Moreover, $SL(2)$ is simply connected with center $\mathbb{Z}/2$, and so $SO(3) = SL(2)/(\mathbb{Z}/2)$ is its adjoint form.
Representations

We next describe the finite dimensional representations of \( \mathfrak{sl}(2) \) (equivalently, of \( \mathcal{U}\mathfrak{sl}(2) \)). Recall that the category \( \text{Rep}_{\text{f.d.}}(\mathfrak{sl}(2)) \) is semisimple, a property which is equivalent to the radical of \( \mathfrak{sl}(2) \) being trivial.

To find the simple/irreducible representations, we think geometrically (think of Lie algebras as algebras of vector fields). Consider the action

\[
SL(2) \to \text{Aut}(\mathbb{A}^2)
\]

By differentiating this we obtain an action

\[
\mathfrak{sl}(2) \to \text{Vect}(\mathbb{A}^2)
\]

given by

\[
H = x\partial_x - y\partial_y
\]

\[
X = y\partial_x
\]

\[
Y = x\partial_y
\]

Therefore \( \mathfrak{sl}(2) \) acts on \( \mathcal{O}(\mathbb{A}^2) \). Since the action is homogeneous, we get a decomposition

\[
\mathcal{O}(\mathbb{A}^2) = \bigoplus_n \mathcal{O}_n(\mathbb{A}^2) = \bigoplus_n \mathbb{C}\langle x^n, x^{n-1}y, \ldots, y^n \rangle = \bigoplus_n L_n
\]

as \( \mathfrak{sl}(2) \)-modules.

**Theorem.** The simples in \( \text{Rep}_{\text{f.d.}}(\mathfrak{sl}(2)) \) are precisely \( L_n \). Moreover, we have \( \dim L_n = n+1 \), \( L_n \cong L_n^* \), and \( L_n \) is the symmetric algebra of the standard representation.

Observe that \( L_0 \) is the trivial representation, \( L_1 \) is the standard representation, and \( L_2 \) is the adjoint representation.

Exercise: The isomorphism \( L_n \cong L_n^* \) (well defined up to a constant) defines a bilinear form which is symmetric for \( n \) even and skew-symmetric for \( n \) odd.

From \( n = 1 \) and \( n = 2 \) this gives us isomorphisms \( \mathfrak{sl}(2) = \mathfrak{sp}(1) \) and \( \mathfrak{sl}(2) = \mathfrak{so}(3) \). In general, you get maps from \( \mathfrak{sl}(2) \) into \( \mathfrak{so}(L_n) \) when \( n \) is even, and \( \mathfrak{sp}(L_n) \) when \( n \) is odd.
Idea of the proof of the theorem: Highest weight analysis.

Let $V$ be finite dimensional, irreducible. Restrict to $\mathbb{C}[H] \subset \mathfrak{u}\mathfrak{s}\mathfrak{l}(2)$. Decompose $V = \bigoplus V_{\lambda}$ where $\lambda$ are the eigenvectors of $H$. We think of the $\lambda$ as belonging to $\text{Spec} \mathbb{C}[H]$. Choose a highest weight $\lambda_{hw}$ (that is, an eigenvalue with maximal real part), and highest weight eigenvector $v_{hw} \in V_{\lambda_{hw}}$.

We have $Xv_{hw} = 0$ and $Yv_{hw} \in V_{\lambda_{hw} - 2}$. One may see that the vectors $v_{hw}, Yv_{hw}, Y^2v_{hw}, \ldots$ span $V$, and have eigenvalues $\lambda_{hw}, \lambda_{hw} - 2, \lambda_{hw} - 4, \ldots$. In $\text{Spec} \mathbb{C}[H]$ we have a picture like this

\[
\begin{array}{cccccc}
\lambda_{hw} & -2k & \ldots & \lambda_{hw} - 2 & \lambda_{hw} \\
\end{array}
\]

Exercise: Show that $V$ is isomorphic to $L_n$ for some $n \geq 0$.

Algebro-geometric interpretation (Borel-Weil-Bott)

Consider the flag variety $\mathbb{P}^1 = (\mathbb{A}^2 \setminus \{0\})/\mathbb{G}_m$. For $n \geq 0$, we have

$$L_n = \mathcal{O}_n(\mathbb{A}^2) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$$

which is what Borel-Weil says for $\mathfrak{s}\mathfrak{l}(2)$.

For $n < -1$, we also have Bott for $\mathfrak{s}\mathfrak{l}(2)$, which says

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^1(n)) = L_{-n-2}$$

You also have

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^1(n)) = 0$$

for negative $n$, and

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^1(n)) = 0$$

for $n \geq -1$. Putting this all together, the cohomology is as follows

\[
\begin{array}{ccccccc}
H^1: & L_1 & L_0 & 0 & 0 & 0 & 0 \\
H^0: & 0 & 0 & 0 & L_0 & L_1 & L_2 \\
n: & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\end{array}
\]

(the symmetry here is to be expected by Serre duality).

Recall that to compute $H^1$ we may use the Cech complex. Write $\mathbb{P}^1 = \mathbb{A}^1 \Pi_{\mathbb{G}_m} \mathbb{A}^1$. Then the following complex computes the cohomology

$$0 \rightarrow \mathbb{C}[x] \oplus \mathbb{C}[y] \rightarrow \mathbb{C}[x, y]/(xy = 1) \rightarrow 0$$

where the middle arrow is defined by $(1, 0) \mapsto 1$ and $(0, 1) \mapsto y^{-n}$.  
What about infinite dimensional representations?

They are very useful, even if you are primarily interested in finite dimensional representations.

Verma Modules

Let $\lambda \in \text{Spec } \mathbb{C}[H]$. Let $B = \mathbb{C}\langle H, X \rangle$ be the subalgebra of $\mathfrak{sl}(2)$ consisting of upper triangular matrices. Let $C_\lambda$ be $\mathbb{C}$ considered as a $B$-module, where $H$ acts by multiplication by $\lambda$, and $X$ acts by zero. The Verma module $V_\lambda$ is then defined as

$$V_\lambda = \mathcal{U}\mathfrak{sl}(2) \otimes_{\mathcal{U}B} C_\lambda$$

This has weights $\lambda, \lambda - 2, \lambda - 4, \ldots$, giving the following picture in $\text{Spec } \mathbb{C}[H]$

\[
\begin{array}{ccc}
\vdots & \lambda - 4 & \lambda - 2 \\
& \lambda & \\
\end{array}
\]

Moreover, $1 \otimes 1 \in V_\lambda$ is the universal highest weight vector of weight $\lambda$. In other words, for any representation $W$ of $\mathfrak{sl}(2)$ we have

$$\text{Hom}_{\mathfrak{sl}(2)}(V_\lambda, W) = \text{Hom}_{\mathcal{U}B}(C_\lambda, W)$$

which is in correspondence with vectors $w$ of $W$ of weight $\lambda$ such that $Xw = 0$.

The Verma modules appear in the BGG (Bernstein, Gelfand, Gelfand) resolution of $L_n$

$$0 \leftarrow L_n \leftarrow V_n \leftarrow V_{-n-2} \leftarrow 0$$

In particular, $V_n$ is not irreducible when $n \in \mathbb{N}$.

Exercise: Show that $V_\lambda$ is irreducible if $\lambda \not\in \mathbb{N}$.

Caution: Not all irreducible representations are generated by higher weight vectors. For an example, consider

$$V = \mathbb{C}\langle \ldots, x^{\lambda-1}y, x^{\lambda}, x^{\lambda+1}y^{-1}, \ldots \rangle$$

where $\lambda \not\in \mathbb{Z}$.

First step in understanding all representations

The Lie algebra $\mathfrak{sl}(2)$ has trivial center, but its enveloping algebra $\mathcal{U}\mathfrak{sl}(2)$ has center $\mathbb{C}[c]$, where $c$ is the Casimir element

$$c = 2XY + 2YX + H^2$$

This element can be thought of as a quantization of the Killing form.

Exercise: $c$ is central.