Jan 19th, 2017

Last time: ($k = \bar{k} = \mathbb{C}$)

Lie alg's (fin. dim'l) $\supseteq$ reductive (ad. rep is semisimple)

$\supseteq$ semisimple (radical = 0)

$\supseteq$ simple (one summand).

Note: ~~"no nontrivial ideals"~~ simple = no nontrivial ideals + nonabelian

~~simple + nonabelian.~~

Ex: $\mathfrak{p} = \left\{ \begin{pmatrix} * & & * \\ & * & \\ 0 & & * \end{pmatrix} \right\}$ is not reductive.

$\mathfrak{g} = \left\{ \begin{pmatrix} * & & 0 \\ & * & \\ 0 & & * \end{pmatrix} \right\} = \Pi \mathfrak{gl}$ is reductive (with nontrivial center)

$= \mathfrak{g}_{s.s} \oplus \mathfrak{z}$

$\mathfrak{g}_{s.s} = \left\{ \begin{pmatrix} tr=0 & & 0 \\ & \square & \\ 0 & & tr=0 \end{pmatrix} \right\}$ semisimple.

$= \bigoplus \mathfrak{sl}_{n_i}$

Lie grp's (fin dim'l) $\supseteq$ complex Lie grp's

$\supseteq$ affine algebraic grps.

$\supseteq$ reductive (trivial unipotent rad)

$\supseteq$ semisimple. (0-dim'l center).

$\supseteq$ simple.

Ex: S^1 - Lie grp. not complex.

Elliptic curve - complex, not algebraic

$\mathbb{G}_m / \mathbb{G}_a$ - algebraic, reductive / not reductive.

$SO(4)$: semisimple, not simple.

$SL(2)$: simple

- Simple Lie Grp's & Algebras.

: Classification of simple lie alg's.

. A_n : $sl(n+1)$. for $n \geq 1$

. B_n : $so(2n+1)$. for $n \geq 2$ ($so(3) \cong sl(2)$)

. C_n : $sp(2n)$. for $n \geq 3$. ($sp(2) = sl(2)$, $sp(4) \cong so(5)$)

. D_n : $so(2n)$. for $n \geq 4$ ($so(2) \cong \mathbb{A}^1$. $so(4) \cong sl(2) \oplus sl(2)$ $so(6) \cong sl(4)$)

Also: E_6. E_7. E_8. [redacted]. G_2. F_4

: Classification of simple lie grp's

. A_n : $SL(n+1) \to \cdots\cdots\cdots \to PGL(n+1)$

[redacted].

coverings

big center : $\mathbb{Z} = \mu_{n+1}$ $\rightsquigarrow$ big π_1 : $\pi_1 = \mathbb{Z}/n+1$

. B_n : $Spin(2n+1) \to SO(2n+1)$

center : $\mathbb{Z}/2\mathbb{Z}$, $\pi_1 = \mathbb{Z}/2\mathbb{Z}$.

. C_n : $Sp(2n) \to PSp(2n)$

center : $\mathbb{Z}/2\mathbb{Z}$, $\pi_1 = \mathbb{Z}/2\mathbb{Z}$.

. D_n : $Spin(2n) \to SO(2n) \to PSO(2n)$

$$\mathbb{Z} = \begin{cases} \mathbb{Z}/4\mathbb{Z} & \text{if } n \text{ odd} \\ \mathbb{Z}/2 \times \mathbb{Z}/2 & \text{if } n \text{ even} \end{cases} \qquad \mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \quad \pi_1 = \mathbb{Z}/2\mathbb{Z} \qquad \pi_1 = \begin{cases} \mathbb{Z}/4\mathbb{Z} & n \text{ odd} \\ \mathbb{Z}/2 \times \mathbb{Z}/2 & n \text{ even} \end{cases}$$

. Exceptional Lie Groups.

- Langland's Duality. (for simple grps).

Combinatorial involution on isom. classes.

NOT functorial ... but "deeply functorial"

Goal: geometric Satake correspondence gives natural construction $G \rightsquigarrow G^\vee$

· Duality descends to Lie algebras:

$A_n \leftrightarrow A_n$

$B_n \leftrightarrow C_n$

$C_n \leftrightarrow B_n$

$D_n \leftrightarrow D_n$

Exceptionals $\leftrightarrow$ Exceptionals.

· preserves direct sums and abelian Lie alg's.

Ex: $\mathfrak{g}$ reductive $= \mathfrak{g}_{ss} \oplus \mathfrak{z} = \bigoplus \mathfrak{g}_i \oplus \mathfrak{z}$.

$\mathfrak{g}^\vee = \bigoplus \mathfrak{g}_i^\vee \oplus \mathfrak{z}^\vee$ ← linear dual.

· For simple grps: involution on lattice of π_1's and centers.

Ex: SL $\leftrightarrow$ PGL.

· Theme:

Rep theory / Harmonic Analysis of A G $\leftrightsquigarrow$ Algebraic geometry of A $G^\vee$

- Abelian Case.

Exercise: T reductive abelian $\Rightarrow T \cong GL(1)^n$.

· Def: T is a torus if $T \cong GL(1)^n$

Notation: character / weight lattice $\mathrm{Hom}(T, \mathbb{G}_m)$

denoted $X^*(T) = \Lambda_T^\vee$ $(\cong \mathbb{Z}^n)$

cocharacter / coweight lattice $\mathrm{Hom}(\mathbb{G}_m, T)$

denoted $X_*(T) = \Lambda_T$ $(\cong \mathbb{Z}^n)$.

Exercise: $T = \mathrm{Spec}(\mathbb{C}[\Lambda_T^\vee]) = \Lambda_T \otimes \mathbb{G}_m$.

· Def: $T^\vee := \mathrm{Spec}\, \mathbb{C}[\Lambda_T] = \Lambda_T^\vee \otimes \mathbb{G}_m$.

Exercise: $\Lambda_{T^\vee}^\vee = \Lambda_T$. $\Lambda_{T^\vee} = \Lambda_T^\vee$

· Geometric Satake for Tori: the following are $\otimes$-equiv.

$$\left(\begin{array}{c}\text{category of} \\ \Lambda_T\text{-graded} \\ \text{vector spaces.}\end{array}\right) \overset{\otimes}{\simeq} \left(\begin{array}{c}\text{category of} \\ \text{fin dim'l reps} \\ \text{of } T^\vee\end{array}\right)$$

also: "convolution" comes from grp structure of Λ_T $\otimes$.

as tensor categories.