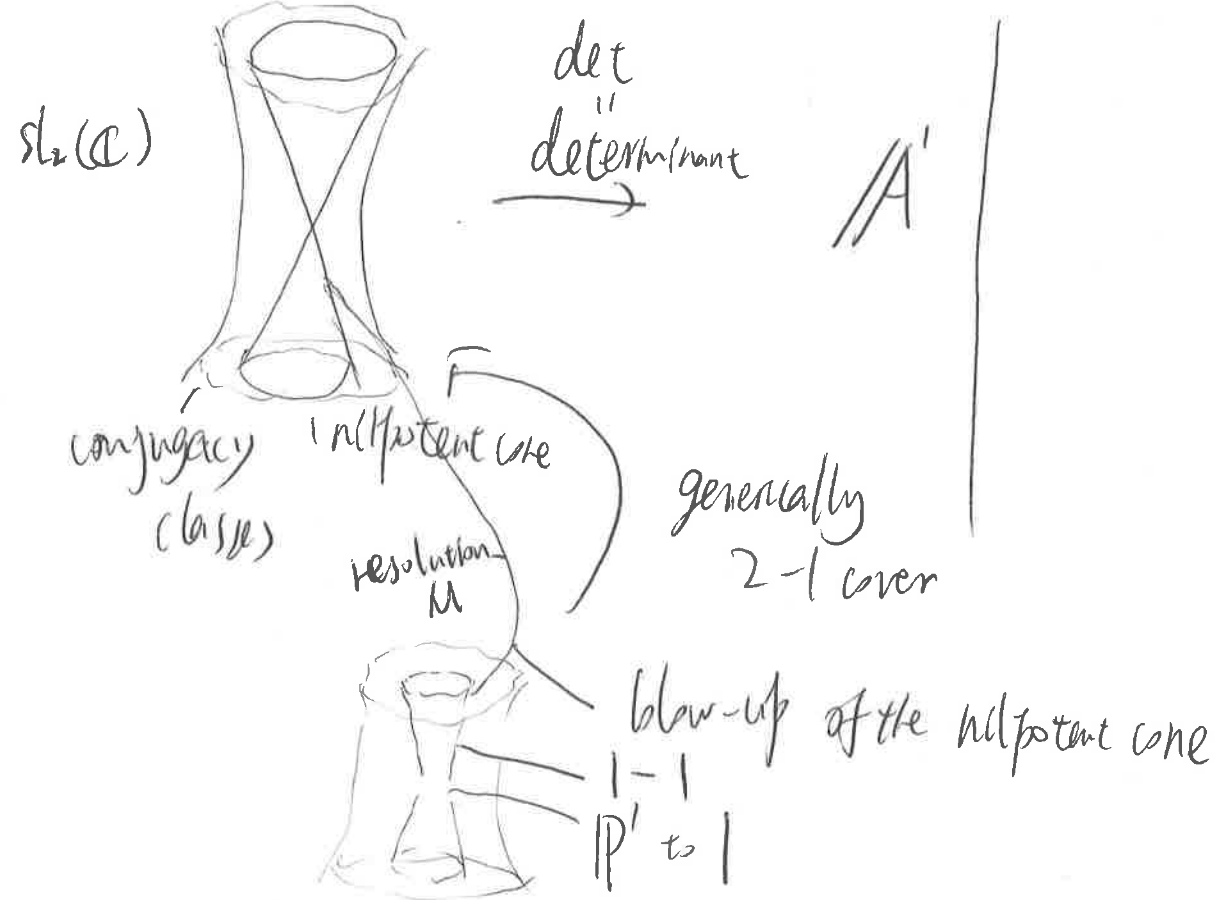
Nadler's class on Langlands Duality

last lecture April 27th, 2017

(1) More details on example $G = GL(2)$, $\lambda = (-1, 0)$
$\mu = (0, 1)$, convolution symmetry

Recall Grothendieck-Springer alteration for $SL(2)$



$\widetilde{sl_2(\mathbb{C})} = \{ (x \in sl(2), l \in \mathbb{P}^1 \text{ s.t. } x \cdot l \subset l \}$

Grothendieck's Springer resolution

$\widetilde{sl_2}(\mathbb{C}) \xrightarrow{\mu} sl_2(\mathbb{C})$

~~regular locus~~

$\mathfrak{sl}_2\mathbb{C}$

$$\begin{tikzcd}
\widetilde{sl(2)}^{reg} \arrow[r, "\mu^{reg}"] \arrow[d, hook, "j"] \arrow[dr, phantom, "2-1"] & sl(2)^{reg} \arrow[r, "\det"] \arrow[d, hook, "j"] & \mathbb{A}^1\setminus\{0\} \arrow[d, hook] \\
\widetilde{sl(2)} \arrow[r, "\mu"] & sl(2) \arrow[r, "\det"] & \mathbb{A}^1
\end{tikzcd}$$

$$\boxed{\mu^{reg}_!\, \mathbb{C}_{\widetilde{sl(2)}^{reg}}[3] = \text{rank 2 local system on } sl(2)^{reg}} \quad \mathcal{L}[3]$$

$$\underset{\substack{\uparrow \\ \text{IC-extension}}}{j_{!*}}\, \mathbb{C}_{\widetilde{sl(2)}^{reg}}[3] \underset{\substack{\uparrow \\ \text{since } \widetilde{sl(2)} \text{ smooth}}}{\simeq} \mathbb{C}_{\widetilde{sl(2)}}[3]$$

Since μ is small, $\mu_!\,\mathbb{C}_{\widetilde{sl(2)}}[3] = \underset{\substack{\uparrow \\ \text{IC-extension}}}{j_{!*}}\,\mathcal{L}[3]$

Conclude $j_{!*}\mathcal{L}[3]|_{\underset{N}{\det=0}} = \mu_!\,\mathbb{C}_{\widetilde{sl(2)}}[3]|_{\underset{N}{\det=0}}$

$$\| $$

$\mu_!\,\mathbb{C}_{\tilde{N}}[3]$ by base change

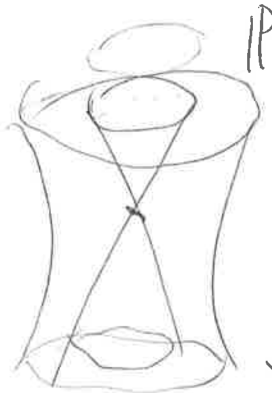
$= (IC_N \oplus \underbrace{IC_0}_{\text{term}})[1]$ coming from $\mathbb{P}^1$ fiber in $\widetilde{sl(2)}$.

by Decomposition Theorem.

Back to convolution example

Claim: $\overline{Gr^{(2),\lambda,\mu}_{GL(2),\mathbb{A}^1}} = sl(2) \times_{\mathbb{A}^1} \mathbb{A}^1$

$$\begin{tikzcd} sl(2) \arrow[dr, "\det"'] & & \mathbb{A}^1 \arrow[dl, "z \to z^2"] \\ & \mathbb{A}^1 & \end{tikzcd}$$



$\overline{Gr^{\lambda+\mu}_{GL(2)}}$ $\qquad Gr^{\lambda}_{GL(2)} \times Gr^{\mu}_{GL(2)} \cong \mathbb{P}^1 \times \mathbb{P}^1$

$$\Big\downarrow \sqrt{\det}$$

$$\mathbb{A}^1$$

Want to calculate IC-sheaf of the family and restrict to special fiber

Recall the small map

$$Gr^{(1),\lambda}_{GL(2)} \tilde{\times} Gr^{(1),\mu}_{GL(2)} \longrightarrow \overline{Gr^{(2),\lambda,\mu}_{GL(2)}}$$

add $\mathbb{P}^1$ at ∞

$\widetilde{\overline{sl(2)}} =$

Since π is small,

$$\pi_! \, IC_{\substack{\text{global convolution}\\ \text{space}}} = IC_{\text{BD Grassmannian}}$$

Restrict to special fibers

$$IC_{\substack{\text{BD Gr}\\ \text{space}}}\Big|_{\varepsilon=0} = \pi_! \, IC_{\overline{\widetilde{N}}} = IC_{\overline{N}} \oplus IC_0$$

(2) Use Geometric Satake to find Canonical Bases for representations

V representation of $G^\vee$, want to find a basis for V with "good" properties:

Recall $V \cong H^*(\overline{Gr^\lambda}, IC^\lambda)$

find basis of cohomology

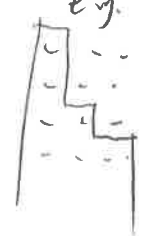
Construct "perverse Schubert basis":

What about ordinary cohomology $H^*(\overline{Gr^\lambda})$?

Construct Schubert basis.

$\underset{\text{Iwahori}}{I} \curvearrowright Gr$ orbits $\leftrightarrow \Lambda_T$ coweights

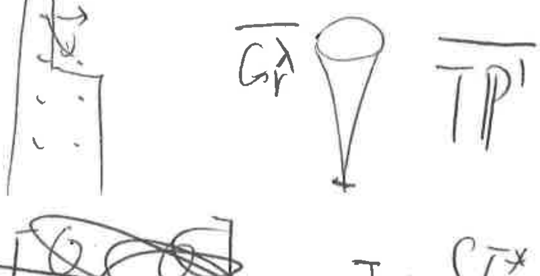
e.g.



Fact: orbits are cells refining $G(\mathcal{O})$ orbits.

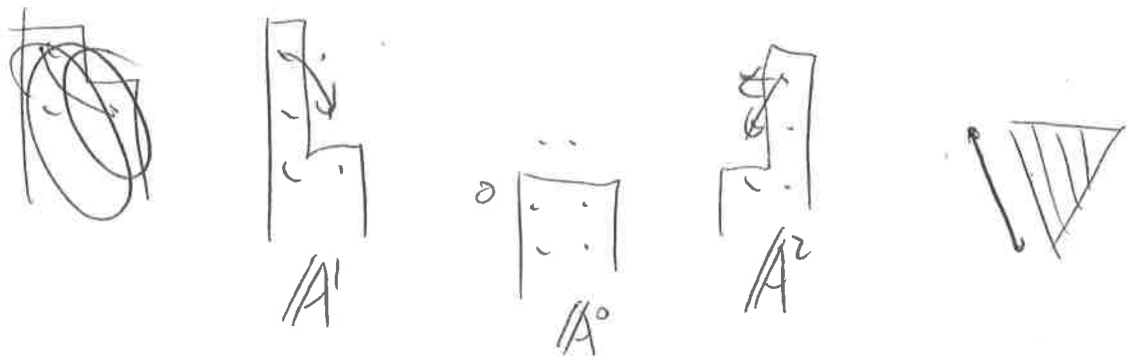
Classes of Iwahori orbits give Schubert basis for $H^*(\overline{Gr^\lambda})$

Ex: (1). $G = SL(2)$, $\lambda = (-1,1)$



$\cancel{I = \begin{bmatrix} \mathcal{O} & \mathcal{O} \\ t\mathcal{O} & \mathcal{O} \end{bmatrix}}$ $\quad I = \left\{ \begin{bmatrix} * & * \\ t* & * \end{bmatrix} \middle| * \in \mathcal{O} \right\}$

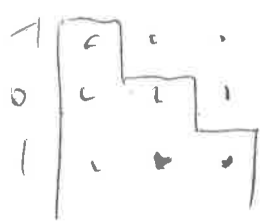
I-orbits in Gr^{λ}



$$H^* \cong \begin{array}{cc} 4 & \mathbb{C} \\ 2 & \mathbb{C} \\ 0 & \mathbb{C} \end{array}$$

with basis given by Iwahori orbits

(2) $G = SL(3)$, $\lambda = (-1, 0, 1)$



line bundle over $SL(3)/B$.

Iwahori orbits

$\mathcal{L}\big|_{B\text{-orbits in } SL(3)/B}$ and base point

$$H^* = \begin{cases} 8 & \mathbb{C} \\ 6 & \mathbb{C}^2 \\ 4 & \mathbb{C}^2 \\ 2 & \mathbb{C} \\ 0 & \mathbb{C} \end{cases}$$

Note $\neq$ IC. (singular)

Now perverse Schubert cells

$N \subset B \subset G$

$\uparrow$ unipotent radical

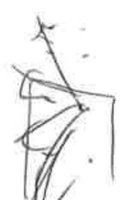
Consider $N(K) \curvearrowright Gr_G$

$$N(K) = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \quad * \in K.$$

$N(K)$ orbits are in bijection with coweight lattice Λ_T

type A

Warning: orbits are neither finite-dimensional or of finite codimension



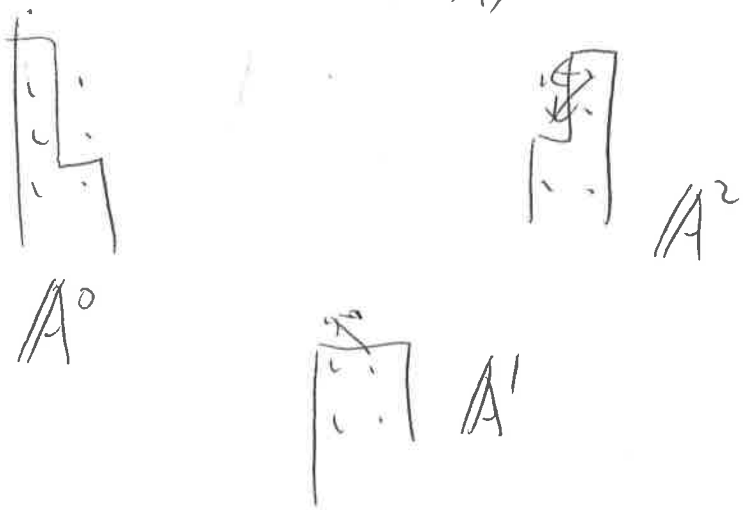
giant contractible spaces

Consider $N(K)$-orbits in Gr intersected with $\overline{Gr^{\lambda}_G}$

Theorem. Each irreducible component of these intersections defines a class in $H^*(\overline{Gr^{\lambda}_G}, IC^{\lambda})$

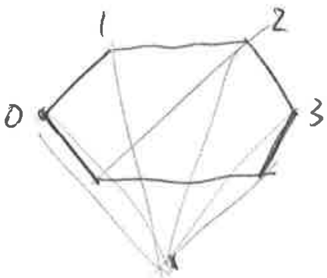
and altogether the irreducible components give a canonical basis.

Ex. (1). $G = SL(2)$, $\lambda = (-1,1)$



compare w Iwahori orbits

(2) $G = SL(3)$, $\lambda = (-1,0,1)$



$H^*(\overline{Gr^\lambda}, IC^\lambda) =$ (= adjoint representation of $PGL(3)$)

8	$\mathbb{C}$
6	$\mathbb{C}^2$
4	$\mathbb{C}^2$
2	$\mathbb{C}^2$
0	$\mathbb{C}$

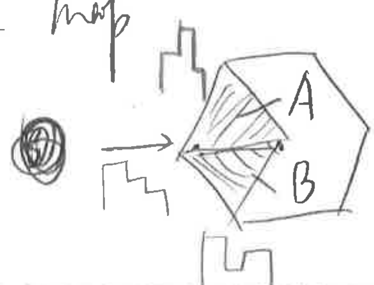
$N(K)$ orbits on $\overline{Gr^\lambda}$

① - 0, 1 dimensional N orbits $SL(3)/B$

② - line bundles over 2, 3 dimensional orbits ~ $SL(3)/B$

③ - two 2-dim irreducible components through base point.

③ Moment map



lattices

Origin of geometry

traditional Satake Isomorphism.

$$\mathbb{C}_c\left(G(O)\backslash G(K)/G(O)\right) \cong \underbrace{\mathbb{C}\left[G(O)\backslash G(K)/N(K)\right] \cong \mathbb{C}[T(K)]}_{\text{Satake transform}}$$