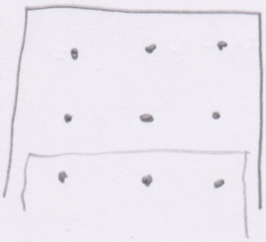
Lattice model of BD grassmannian

$G = GL(n)$, fix $N \gg 0$

$$Gr^N_{GL(n)} = \{ \underbrace{t^N \mathcal{O}^n \subseteq L \subseteq t^{-N} \mathcal{O}^n}_{\mathcal{O}\text{-submodules}} \} = \{ \mathcal{O}\text{-submodules of } \underbrace{t^{-N}\mathcal{O}^n / t^N \mathcal{O}^n}_{\text{finite dimensional}} \}$$



Fix some curve C, recall

$$Gr^{(k)}_{G,C} = \{ (x_1, \dots, x_k) \in C^k,\ G\text{-bundle } \mathcal{P} \text{ and triv } \mathcal{P}^0 \xrightarrow{\simeq} \mathcal{P}|_{C \setminus \cup x_i} \}$$

Factorization $Gr^{(k)}_{G,C} \Big|_{l \text{ distinct points}} \simeq \prod^l Gr_C$

Lattice model for $G = GL(n)$

$$Gr_{GL(n),C} = \{ (x_1, \dots, x_k) \in C^k,\ \mathcal{O}^n_C(-N \textstyle\sum x_i) \underset{\mathcal{O}_C\text{-submodule}}{\subseteq} \mathcal{L} \subseteq \mathcal{O}^n_C(N \textstyle\sum x_i) \}$$

$$\simeq \{ (x_1, \dots, x_k) \in C^k,\ \mathcal{L} \underset{\mathcal{O}_C\text{-submodule}}{\subseteq} \underbrace{\mathcal{O}^n_C(N \textstyle\sum x_i) / \mathcal{O}^n_C(-N \textstyle\sum x_i)}_{\text{torsion sheaf supported at } \cup x_i} \}$$

Reformulate convolution product in an evidently symmetric form

$$G(O)\backslash Gr_G \times G(O)\backslash Gr_G \xleftarrow{p_1 \times p_2} G(O)\backslash Gr(K) \overset{G(O)}{\times} Gr_G \xrightarrow{m} G(O)\backslash Gr_G$$

Moduli interpretation

disk

$$G(O)\backslash Gr = \{ \mathcal{P}_1, \mathcal{P}_2 \text{ G-bundles on } D, \ \mathcal{P}_1|_{D^\times} \simeq \mathcal{P}_2|_{D^\times} \}$$

Convolution diagram

$$\left\{ \begin{matrix} \mathcal{P}_1, \mathcal{P}_2 \text{ on } D \\ \mathcal{P}_1|_{D^\times} \simeq \mathcal{P}_2|_{D^\times} \end{matrix} \right\} \times \left\{ \begin{matrix} \mathcal{P}_3, \mathcal{P}_4 \text{ on } D \\ \mathcal{P}_3|_{D^\times} \simeq \mathcal{P}_4|_{D^\times} \end{matrix} \right\} \longleftarrow \left\{ \begin{matrix} \mathcal{P}_a, \mathcal{P}_b, \mathcal{P}_c \text{ on } D \\ \mathcal{P}_a|_{D^\times} \simeq \mathcal{P}_b|_{D^\times} \simeq \mathcal{P}_c|_{D^\times} \end{matrix} \right\}$$

$$\downarrow$$

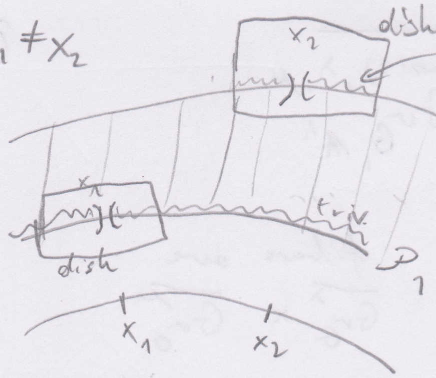
$$\left\{ \begin{matrix} \mathcal{P}_a, \mathcal{P}_c \text{ on } D \\ \mathcal{P}_a|_{D^\times} \simeq \mathcal{P}_c|_{D^\times} \end{matrix} \right\}$$

Replace D with curve C

$$Gr^{(1)}_{G,C} * Gr^{(1)}_{G,C} \qquad\qquad Gr^{(1)}_{G,C}$$

$$\left\{ \begin{matrix} x_1, x_2 \in C, \ \mathcal{P}_1, \mathcal{P}_2 \text{ G-bundles} \\ \mathcal{P}^\circ \simeq \mathcal{P}_1|_{C \setminus x_1}, \ \mathcal{P}_1|_{C\setminus x_1} \simeq \mathcal{P}_2|_{C \setminus x_2} \end{matrix} \right\} \qquad \left\{ \begin{matrix} x \in C, \ \mathcal{P} : \text{G-bundle} \\ \mathcal{P}^\circ \simeq \mathcal{P}|_{C \setminus x} \end{matrix} \right\}$$

1° $x_1 \neq x_2$

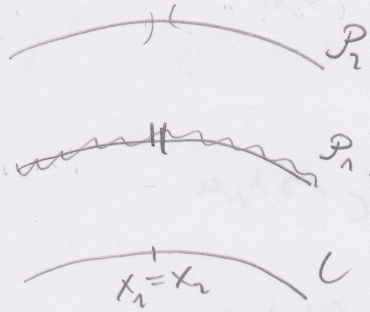


trivialization induced from trivialization on $\mathcal{P}_1$ by an isomorphism

isom to $Gr^{(2)}_{G,C}\big|$ 2 distinct points

fibers isomorphic to $Gr \times Gr$

2° $x_1 = x_2$



isomorphic to $Gr_G * Gr_G$ (middle term of any diagram $G(K) \overset{G(O)}{\times} Gr_G$)

distinct pts — fixed equal pts

$$\begin{tikzcd}
\text{restrict.} \arrow[r, phantom, "\subset"] \arrow[d, "\wr"] & Gr^{(1)}_{G,C} * Gr^{(1)}_{G,C} \arrow[r, phantom, "\supset"] \arrow[d] & Gr_G * Gr_G \arrow[d] \\
\text{restrict.} \arrow[r, phantom, "\subset"] \arrow[dr, no head] & Gr^{(2)}_{G,C} \arrow[r, phantom, "\supseteq"] & Gr_G \arrow[dl, no head] \\
& C^2 &
\end{tikzcd}$$

Now let's consider IC^λ, IC^μ

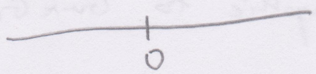
$$\overline{Gr}^{\lambda}_G * \overline{Gr}^{\mu}_G \xrightarrow{m} \overline{Gr}^{\lambda+\mu}_G$$

$$IC^{\lambda,\mu} \mapsto m_! IC^{\lambda,\mu}$$

For simplicity fix $C = \mathbb{A}^1$, $x_1 = -x_2$

$$\overline{Gr}^{(1),\lambda}_{G,\mathbb{A}^1} * \overline{Gr}^{(1),\mu}_{G,\mathbb{A}^1}\Big|_{t\neq 0} \simeq \overline{Gr}^{(2),\lambda,\mu}_{G,\mathbb{A}^1}$$

fibers are $\overline{Gr}^{\lambda}_G \times \overline{Gr}^{\mu}_G$



$\mathbb{A}^1 = \operatorname{Spec}(\mathbb{C}[t])$

Work $IC^{(2),\lambda,\mu}$ for IC sheaf of $\overline{Gr}^{(2),\lambda,\mu}_{G,\mathbb{A}^1}$.

Extend to IC sheaf

$$\begin{array}{ccc} t\neq 0 \subset \overline{Gr}^{(1),\lambda}_{G,\mathbb{C}} * \overline{Gr}^{(1),\mu}_{G,\mathbb{C}} & & \widetilde{IC}^{\lambda,\mu} \\ \downarrow{\scriptstyle 2} \qquad\qquad \downarrow & & \\ t\neq 0 \subset \overline{Gr}^{(2),\lambda,\mu}_{G,\mathbb{A}^1} & & IC^{(2),\lambda,\mu} \end{array}$$

Facts

1) m is small, so $m_!\widetilde{IC}^{\lambda,\mu} = IC^{(2),\lambda,\mu}$

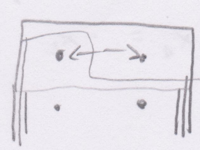
2) $\widetilde{IC}^{\lambda,\mu}\big|_{t=0} \simeq IC^{\lambda,\mu}$

Upshot: $IC^{\lambda} * IC^{\mu} = IC^{(2),\lambda,\mu}\Big|_{t=0}$

Example

$G = GL(2)$ $\quad \mu = (0,-1), \ \lambda = (-1,0)$

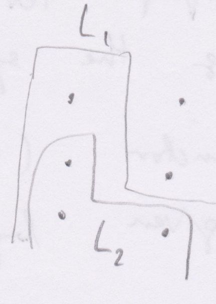
$Gr^{\lambda}_G = \overline{Gr}^{\lambda}_G = \mathbb{P}^1$



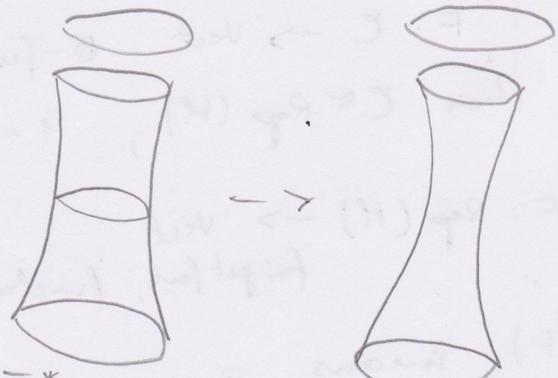
Convolution

$$\begin{tikzcd} Gr_G^{\lambda} * Gr_G^{\mu} \arrow[d, "m"] = \mathbb{P}^1 * \mathbb{P}^1 \\ \overline{Gr_G^{\lambda+\mu}} = \text{forget } L_2 \end{tikzcd}$$

$$\mathbb{P}^1 = L_1, \quad \mathbb{P}^1 \ni L_2$$



$$\begin{tikzcd} \overline{T}^*\mathbb{P}^1 \arrow[d] \\ \overline{Q}^2 \end{tikzcd}$$



$\overline{T}^*\mathbb{P}^1$

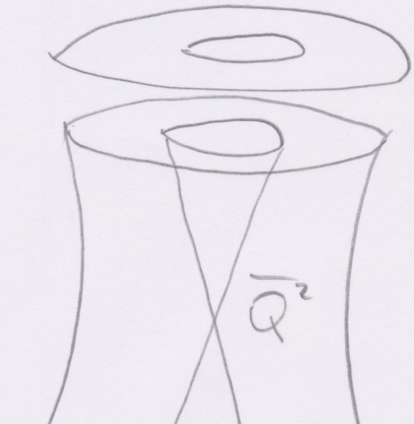
$\overline{Q}^2$

$$\begin{tikzcd} \overline{Gr^{(2),\lambda,\mu}_{G,\mathbb{A}^1}} \quad \mathbb{P}^1\times\mathbb{P}^1 \arrow[r] \arrow[d] & \overline{Q}^2 \arrow[d] \\ t\neq 0 & t=0 \end{tikzcd}$$

$\longrightarrow \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[t])$

$x^2+y^2+z^2=t^2$

degeneration of quadrics



$\overline{Q}^2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$

(degeneration)

We can equip $\mathrm{Perv}^{G(O)}(\mathrm{Gr}_G)$ with $\otimes$-category structure using the symmetry exhibited.

Fiber functor (= faithful $\otimes$-functor to Vect) given by global sections

Tannakian

$\mathcal{C}$ $\otimes$-cat, $F: \mathcal{C} \to \mathrm{Vect}$ $\otimes$-funct.,

then $\mathcal{C} \simeq \mathrm{Rep}(H)$, $H = \mathrm{Aut}^{\otimes}(F)$

<u>Ex</u> $\mathcal{C} = \mathrm{Rep}(H)$, $F: \mathrm{Rep}(H) \to \mathrm{Vect}$ forgetful functor

$\varphi \in \mathrm{Aut}^{\otimes}(F)$ means

$\forall V$ $\varphi_V : F(V) \xrightarrow{\sim} F(V)$

$\varphi_{V \otimes W} \simeq \varphi_V \otimes \varphi_W$

so $\mathrm{Aut}^{\otimes}(F) = H$.