Apr. 25th: 2017

· More Scientific view of the lattice model of BD-Grass.

$G = GL_n$. fix $N \gg 0$.

$$Gr_G^N = \{ t^N \mathcal{O}^n \subset L \subset t^{-N} \mathcal{O}^n \}$$

$$= \{ \mathcal{O}\text{-submod of } t^{-N}\mathcal{O}^n / t^N \mathcal{O}^n \}$$

Fix some curve C. recall

$$Gr_{G.C}^{(k)} = \left\{ (x_1, \dots, x_k) \in C^k ; \begin{array}{c} G\text{-bun } P. \text{ and} \\ \text{trivialization } P^0 \xrightarrow{\sim} P|_{C \setminus \underline{x}} \end{array} \right\}$$

factorization: $Gr_{G.C}^{(k)}\Big|_{\overset{\circ}{C}} = \prod^k Gr_G$.

Lattice model: fix $N \gg 0$.

$$Gr_{G.C}^{(k).N} = \{ \underline{x} \in C^k . \ \underset{\substack{\uparrow \\ \text{allow 0's} \\ \text{of order N at} \\ x_i}}{\mathcal{O}_C^n(-N\textstyle\sum x_i)} \subset \overset{\substack{\mathcal{O}\text{-submod.} \\ \downarrow}}{\mathcal{L}} \subset \underset{\substack{\uparrow \\ \text{allow poles} \\ \text{of order N} \\ \text{at } x_i}}{\mathcal{O}_C^n(N\textstyle\sum x_i)} \}$$

$$= \{ \underline{x} \in C^k ; \ \underset{\substack{\uparrow \\ \mathcal{O}\text{-submod}}}{\mathcal{L}} \subset \underset{\substack{\uparrow \\ \text{torsion sheaf} \\ \text{(set) supp at } \cup x_i.}}{\frac{\mathcal{O}_C^n(N\sum x_i)}{\mathcal{O}_C^n(-N\sum x_i)}} \ \mathcal{O}\text{-mod} \}.$$

· Reformulate convol'n in a symmetric way.

Before:

$$G(\mathcal{O})\backslash Gr \times G(\mathcal{O})\backslash Gr \xleftarrow{p_1 \times p_2} G(\mathcal{O})\backslash G(K) \overset{G(\mathcal{O})}{\times} Gr \xrightarrow{m} G(\mathcal{O})\backslash Gr$$

Moduli interpretation:

$$G(\mathcal{O})\backslash G(K) = \{ P_1, P_2 \text{ G-bun on } D,\ P_1|_{D^\times} \cong P_2|_{D^\times} \}.$$

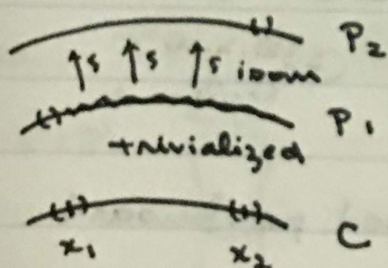
$$\{P_1, P_2\} \times \{P_3, P_4\} \leftarrow \left\{ \begin{matrix} P_a, P_b, P_c, \text{ on } D \\ P_a|_{D^\times} \cong P_b|_{D^\times} \cong P_c|_{D^\times} \end{matrix} \right\} \rightarrow \left\{ \begin{matrix} P_a, P_c \text{ on } D \\ P_a|_{D^\times} \cong P_c|_{D^\times} \end{matrix} \right\}$$

a b b c.

Replace D with curve C

$$\begin{array}{ccc} Gr^{(1)}_{G,C} * Gr^{(1)}_{G,C} & \longrightarrow & Gr^{(1)}_{G,C} \\ \| & & \| \\ \{x_1, x_2 \in C,\ P_1, P_2, & & \{x \in C,\ P \text{ a G-bun} \\ P^0 \xrightarrow{\sim} P_1|_{C \setminus x_1}, \text{ and} & & P^0 \cong P|_{C \setminus x} \}. \\ P_1|_{C \setminus x_2} \xrightarrow{\sim} P_2|_{C \setminus x_2} \}. & & \end{array}$$

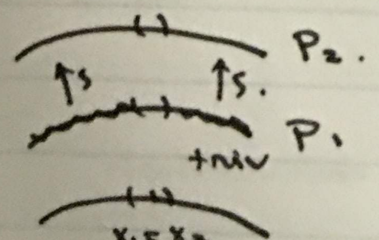
$x_1 \neq x_2$.



isom $Gr^{(2)}_{G,C}|_{2 \text{ distinct pts}}$

fibers isom to $Gr \times Gr$.

$x_1 = x_2$.



isom to $Gr_G * Gr_G$

(middle term of convol'n diagram $G(K) \overset{G(\mathcal{O})}{\times} Gr_G$)

Distinct $x_1 \neq x_2$ $\qquad\qquad\qquad\qquad$ $x_1 = x_2$

$$\begin{array}{ccccc} \text{Restr.} & \subseteq & Gr^{(1)}_{G,C} * Gr^{(1)}_{G,C} & \supseteq & \text{Restr} \\ \downarrow s & & \downarrow & & \downarrow \text{convolve.} \\ \text{Restr.} & \subseteq & Gr^{(2)}_{G,C} & \supseteq & \text{Restr.} \end{array}$$

Let's compute $IC^\lambda * IC^\mu$.

Usually: $\overline{Gr^\lambda_G} \times \overline{Gr^\mu_G} \xrightarrow{m} \overline{Gr^{\lambda+\mu}_G}$

$$IC^{\lambda,\mu} \longmapsto m_! IC^{\lambda,\mu}.$$

For simplicity, $C = \mathbb{A}^1$, $x_1 = -x_2 = t$.

$$Gr^{(1),\lambda}_{G,\mathbb{A}^1} * Gr^{(1),\mu}_{G,\mathbb{A}^1} \Big|_{t \neq 0}$$

$$\simeq \overline{Gr^{(2),\lambda,\mu}_{G,\mathbb{A}^1}}. \qquad \text{fibers} = \overline{Gr^\lambda} \times \overline{Gr^\mu}$$

Write $IC^{(2),\lambda,\mu}$ for IC sheaf of $Gr^{(2),\lambda,\mu}_{G,\mathbb{A}^1}$.

Extend to IC sheaf using truncated pushforward.

$$\begin{array}{ccccc} t \neq 0 & \subset & \overline{Gr^{(1),\lambda}} * \overline{Gr^{(1),\mu}} & & \widetilde{IC}^{\lambda,\mu}. \\ \downarrow s & & \downarrow m & & \\ t \neq 0 & & \overline{Gr^{(2)\lambda,\mu}} & & IC^{(2),\lambda,\mu}. \end{array}$$

Fact: 1) m is small, so

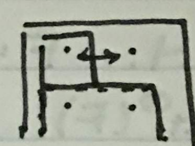
$$m_! \widetilde{IC}^{\lambda,\mu} = IC^{(2),\lambda,\mu}.$$

2). $\widetilde{IC}^{\lambda,\mu}|_{t=0} \simeq IC^{\lambda,\mu}.$

Upshot: $IC^\lambda * IC^\mu = IC^{(2),\lambda,\mu}|_{t=0}.$

Example: $G = GL_2$, $\lambda = (-1,0)$, $\mu = (0,-1)$.

$$Gr^\lambda_G = \overline{Gr}^\lambda_G = \mathbb{P}^1$$



same for μ.

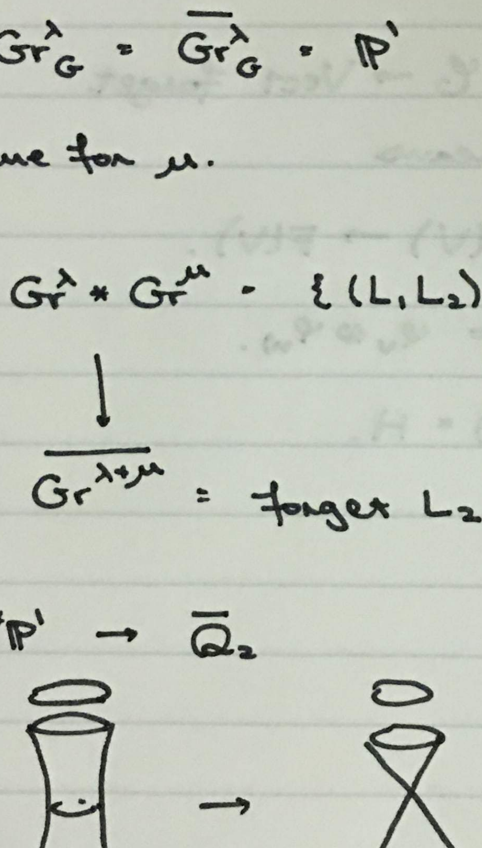
Convolu $Gr^\lambda * Gr^\mu = \{(L_1, L_2) : \mathrm{relpos}(L_2, \mathcal{O}^2) = \lambda,\ \mathrm{relpos}(L_2, L_1) = \mu\}.$

$$\downarrow$$

$\overline{Gr}^{\lambda+\mu}$ = forget L_2.

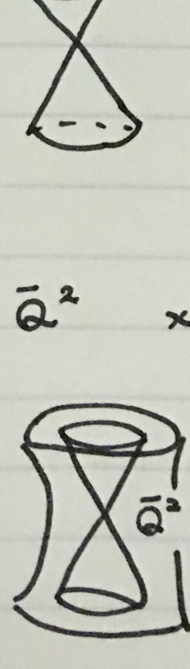
Picture $T^*\mathbb{P}^1 \to \overline{Q}_2$

$\mathbb{P}^1 * \mathbb{P}^1$



$\overline{Gr}^{(2),\lambda,\mu}_{G,\mathbb{A}^1}$ $\quad \mathbb{P}^1 \times \mathbb{P}^1 \rightsquigarrow \overline{Q}^2 \quad x^2+y^2+z^2=t^2.$

degeneration



$\overline{Q}^2_{\text{nonsing}} \simeq \mathbb{P}^1 \times \mathbb{P}^1.$

We've equipped $\mathrm{Perv}^{G(O)}(\mathrm{Gr}_G)$ with $\otimes$-cat structure with the symmetry exhibited.

$\exists$ fiber functor (faithful $\otimes$-functor) $\to$ Vect given by global sect.

Tannakian formalism: $\mathcal{C}$ a $\otimes$-cat, $F : \mathcal{C} \to \mathrm{Vect}$ $\otimes$-functor, then $\mathcal{C} \simeq \mathrm{Rep}(H)$. where

$$H = \mathrm{Aut}^{\otimes}(F).$$

Ex: $\mathcal{C} = \mathrm{Rep}\, H$. $F : \mathcal{C} \to \mathrm{Vect}$ forget.

$\varphi \in \mathrm{Aut}^{\otimes}(F)$ means

$$\varphi_V : F(V) \to F(V).$$

$$\varphi_{V \otimes W} = \varphi_V \otimes \varphi_W.$$

check $\mathrm{Aut}^{\otimes}(F) = H$.