We begin with a more scientific view of the lattice model of the 8D Gaussian $G = GL(n)$. Fix $N > 0$. The affine Gaussian contains the finite piece:

$$G_N = \{ \mathbf{C}^N \subset \mathbf{C} \times \mathbf{C}^N \}$$

$O$-submodule $\{ 0 \}$-submodule of $\mathbf{C}^N \times \mathbf{C}^N$.

Now fix a smooth curve $C$.

Recall $Gr_{c, 0} = \{ (x_1, \ldots, x_k) \in C^k \mid P \text{ is stable, } P^0 \rightarrow P \text{ is a vector bundle} \}$.

It has stratification $Gr_{c, 0} \rightarrow \mathcal{M}^g C^k$.

Let $L$ be a lattice model $GL(n)$-bundle $\mathcal{E}$.

Then as above we have the finite piece $Gr_{c, 0} \subset \mathbf{C}^k \mathbf{C}^k$.

$$\{ (x_1, \ldots, x_k) \in C^k \mid Q(x_1, \ldots, x_k) \subset L \subset Q(x_1, \ldots, x_k) \}$$

Goal: Reformulate convolution product in an evidently geometric form.

We can write:

$$\text{Conv. Diagram: } \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E} \times \mathcal{E}$$

Replacing $\mathcal{E}$ with a curve $C$.

$$\mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E} \times \mathcal{E}$$

$$\mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E} \times \mathcal{E}$$

$$\mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E} \times \mathcal{E}$$

What is $\mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E} \times \mathcal{E}$? If $x_1 \neq x_2$, we have a 6-bundle $P$, twisted off $x_1$.

Hence it also has a trivialization $\mathcal{E}$ is a trivial bundle over $\mathbb{C}$ and $x$.

Anyway, we see that in this case we are getting:

$$\mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E} \times \mathcal{E}$$

$$\mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E} \times \mathcal{E}$$

If $x_1 = x_2$, then $\mathcal{E}$ is a trivial $\mathcal{E}$.

Finally:

$$\mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E} \times \mathcal{E}$$

$$\mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E} \times \mathcal{E}$$

$$\mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E} \times \mathcal{E}$$

$$\mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E} \times \mathcal{E}$$

$$\mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E} \times \mathcal{E}$$
Update: We have a map \( \text{Gr}_0,1 \times \text{Gr}_0,1 \to \text{Gr}_0,1 \) whose fiber
on \( x, x_{12} \) is \( \text{Gr}_0 \times \text{Gr}_0 \to \text{Gr}_0 \times \text{Gr}_0 \)
on \( x = x_{12} \) is \( \text{Gr}_0 + \text{Gr}_0 \to \text{Gr}_0 \), which is the pushforward in the comm. diag.

Now let's use this to see why convolution is commutative.
To compute \( \text{const} \times \text{conv} \) of \( IC^3 \) with \( IC^m \), we've been
studying \( \frac{IC^3 \times IC^m}{IC^3, IC^m} \leftarrow \frac{IC^3 \times IC^m}{IC^3, IC^m} \)

For simplicity, fix \( C = \mathbb{A}' \), \( \mathbb{A}' \neq \mathbb{A} \), \( x = -x_{12} \)
the \( C \text{-} \text{Spec} \) \( C = \mathbb{A}' \)

At \( t \neq 0 \), we have \( \text{Gr}_0,1 \times \text{Gr}_0,1, 1 \to \text{Gr}_0,1, 1 \), whose fibers are \( \text{Gr}_0 + \text{Gr}_0 \).
Write \( IC^3, IC^m \) for the \( IC \) sheaf of \( \text{Gr}_0,1 \times \text{Gr}_0,1 \), whose fibers are \( \text{Gr}_0 + \text{Gr}_0 \).
Extend this to \( IC \) sheaves on \( \text{Gr}_0,1 \times \text{Gr}_0,1 \) with \( IC^3 \)
i.e., we're extending the \( IC \) sheaf using the inclusion of \( t = 2 \) pieces to the above and the whole thing.

Fact: \( m \) is small, so \( m IC^m = IC^3, IC^m \)

1. \( IC^3, IC^m \leftarrow IC^3, IC^m \)
So we've found a symmetric way of describing the convolution product.

Ex: \( G = GL(2), \lambda = (-1,0), m = (0,-1) \), \( \text{Gr}_0,1 \times \text{Gr}_0,1 \), \( \text{Gr}_0,1 = \mathbb{P}^1 \), semi-\( m \).

Convolution \( \text{Gr}_0,1 \times \text{Gr}_0,1 = \mathbb{P}^1 \times \mathbb{P}^1 \)

This morphism gives \( \mathbb{P}^1 \times \mathbb{P}^1 \to Q_2 \)

BD points \( \text{Gr}_0,1, IC^3, IC^m \), \( \text{Proj} IC^m \to Q_2 \), \( \lambda^2 + \gamma^2 \), \( t^2 \) elements of \( \mathbb{Q} \).

Thus \( \text{Proj} IC^m \to Q_2 \), \( t \neq 0 \), \( t = 0 \), \( \text{Spec} \) \( t \)
So we have two ways of relating 2 copies of P' to \(\overline{Q}^2\):
- convolution involves a map \(P' \times P' \rightarrow \overline{Q}^2\)
- or we can study a degeneration of \(P' \times P'\) to \(\overline{Q}^2\).

Now we're ready to finish the Tannakian dictionary which will prove the geometric Satake theorem.

We have seen that the symmetry discussed above allows us to equip \(\text{Rep}_G(\text{Gr}_0)\) with the structure of a \(\infty\)-cat.

The fibre fund (faithful \(\infty\)-functor to Vect) is given by \(\text{globalsector}\).

Tannakian Recollection: \(C\ \text{\(\infty\)}\text{-cat}, F : C \rightarrow \text{Vect}\ \text{\(\infty\)}\text{-functor} \Rightarrow C = \text{Rep}(H)\) when \(H = \text{Aut}^0(F)\).

\(\text{Eq: } C = \text{Rep}(H), F : \text{Rep}(H) \rightarrow \text{Vect}\) \(\text{\(\infty\)}\text{-functor}\). Then \(\forall \psi \in \text{Aut}^\circ(F)\)

means for \(v \in \text{Rep}(H)\), we have \(\psi : F(v) \cong F(w)\), with \(\psi \circ \phi \cong \phi \circ \psi\),

so \(\text{Aut}^\circ(F) = H\).

Applying this theory to \(\text{Rep}_G(\text{Gr}_0)\), we obtain an alg. group, which you can check is the combinatorially defined dual group.