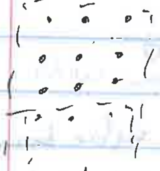
25 April

We begin with a more scientific view of the lattice model of the BD Grassmannian $G = GL(n)$. Fix $N >> 0$. The affine Grassmannian contains a finite piece
$Gr^N_{GL(n)} = \{ t^N \mathcal{O}^n \subset L \subset t^{-N}\mathcal{O}^n \quad \mathcal{O}\text{-submodule} \} = \{ \mathcal{O}\text{-submodules of } t^{-N}\mathcal{O}^n / t^N \mathcal{O}^n \}$



Now fix a smooth curve C.

Recall $Gr^{(k)}_{G,C} = \{ (x_1, \ldots, x_k) \in C^k,\ P \text{ G-bundle},\ P^0 \xrightarrow{\sim} P|_{X - \cup x_i} \text{ triv} \}$

It has factorization $Gr^{(k)}_{G,C}|_{\ell \text{ distinct pts}} \simeq \prod^{\ell} Gr_G$

<u>Lattice model for $G = GL(n)$</u>: Fix $N >> 0$. Then as above we have a finite piece $Gr^{(k),N}_{GL(n),C} = \{ (x_1, \ldots, x_k) \in C^k,\ \mathcal{O}^n_C(-N \cdot \Sigma x_i) \subset \mathcal{L} \subset \mathcal{O}^n_C(\Sigma x_i) \}$ ← $\mathcal{O}_C$-submodule

$= \{ (x_1, \ldots, x_k) \in C^k,\ \underset{\mathcal{O}_C\text{-submod}}{\mathcal{L}} \subset \underbrace{\mathcal{O}^n_C(N \Sigma x_i) / \mathcal{O}^n_C(-N\Sigma x_i)} \}$

<u>NB</u>: This is a torsion sheaf (set-theoretically) supported at $\cup x_i$

<u>Goal</u>: Reformulate convolution product in an evidently symmetric form.

convolution diagram: $G(\mathcal{O}) \backslash Gr_G \times G(\mathcal{O}) \backslash Gr_G \xleftarrow{p_1 \times p_2} G(\mathcal{O}) \backslash G(\mathcal{K}) \overset{G(\mathcal{O})}{\times} Gr_G \xrightarrow{m} G(\mathcal{O}) \backslash Gr_G$

<u>Moduli interpretation</u>: $G(\mathcal{O}) \backslash Gr_G = \{ P_1, P_2 \text{ G-bundles on } D,\ P_1|_{D^\times} \xrightarrow{\sim} P_2|_{D^\times} \}$

conv. diagram: $\left\{ \begin{matrix} P_1, P_2 \text{ on } D \\ \underset{a}{P_1}|_{D^\times} \simeq \underset{b}{P_2}|_{D^\times} \end{matrix} \right\} \times \left\{ \begin{matrix} P_3, P_4 \text{ on } D \\ \underset{b}{P_3}|_{D^\times} \simeq \underset{c}{P_4}|_{D^\times} \end{matrix} \right\} \leftarrow \left\{ \begin{matrix} P_a, P_b, P_c \text{ on } D \\ P_a|_{D^\times} \simeq P_b|_{D^\times} \simeq P_c|_{D^\times} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} P_a, P_c \text{ on } D \\ P_a|_{D^\times} \simeq P_b|_{D^\times} \end{matrix} \right\}$

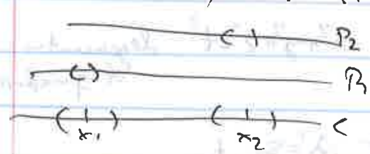
~~We could [illegible] getting rid of the $G(\mathcal{O})$ quotient on the left in each piece of the diagram,~~
~~$Gr_G \times G(\mathcal{O}) \backslash Gr_G \leftarrow G(\mathcal{K}) \overset{G(\mathcal{O})}{\times} Gr_G \rightarrow Gr_G$~~

We'd like ~~[illegible]~~ global analogue of the right ; middle piece of this convolution diagram, replacing the disk D with a curve C.

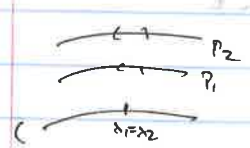
$Gr^{(1)}_{G,C} * Gr^{(1)}_{G,C} \longrightarrow Gr^{(1)}_{G,C}$

$\left\{ \begin{matrix} (x_1, x_2) \in C^2,\ P_1, P_2 \text{ G-bundles}, \\ P_0 \simeq P_1|_{C - x_1},\ P_1|_{C - x_2} \simeq P_2|_{C - x_2} \end{matrix} \right\} \longrightarrow \{ x \in C,\ P \text{ G-bundle},\ P^0 \xrightarrow{\sim} P|_{C - x} \}$

What is $Gr^{(1)}_{G,C} * Gr^{(1)}_{G,C}$? If <u>$x_1 \neq x_2$</u>: We have a G-bundle P_1, trivialized off $\{x_1\}$, ~~[illegible]~~ the moduli space of which is Gr_G, and then we have P_2, which agrees with P_1 away from x_2; hence it also has a trivialization in a formal punctured disk around x.



Anyway, we see that in this case we are getting the fiber $Gr^{(2)}_{G,C}|_{(x_1, x_2)} \simeq Gr \times Gr$



If <u>$x_1 = x_2$</u>: this is isomorphic to ~~[illegible]~~ $Gr_G * Gr_G$ (the middle term $G(\mathcal{K}) \overset{G(\mathcal{O})}{\times} Gr_G$ of the conv. diagram).

Upshot: We have a map $\mathrm{Gr}^{(1)}_{G,C} * \mathrm{Gr}^{(1)}_{G,C} \longrightarrow \mathrm{Gr}^{(2)}_{G,C}$ whose fiber

on $x_1 \neq x_2$ is $\mathrm{Gr}_G \times \mathrm{Gr}_G \xrightarrow{\sim} \mathrm{Gr}_G \times \mathrm{Gr}_G$;

on $x_1 = x_2$ is $\mathrm{Gr}_G * \mathrm{Gr}_G \longrightarrow \mathrm{Gr}_G$, which is the pushfwd arrow in the conv. diagram.

Now let's use this to see why convolution is commutative.

To compute the convol. of IC^λ with IC^μ, we've been studying $\overline{\mathrm{Gr}_G^\lambda} \times \overline{\mathrm{Gr}_G^\mu} \xrightarrow{m} \mathrm{Gr}_G^{\lambda+\mu}$

$IC^{\lambda,\mu} \longmapsto m_! IC^{\lambda\mu}$

For simplicity, fix $C = \mathbb{A}^1$, $x_1 = -x_2$. $\mathbb{A}^1 = \mathrm{Spec}\, \mathbb{C}[t]$

At $t \neq 0$, we have $\overline{\mathrm{Gr}^{(1),\lambda}_{G,\mathbb{A}^1}} * \overline{\mathrm{Gr}^{(1),\mu}_{G,\mathbb{A}^1}}\big|_t \cong \overline{\mathrm{Gr}^{(2),\lambda\mu}_{G,\mathbb{A}^1}}$, whose fibers are $\overline{\mathrm{Gr}_G^\lambda} \times \overline{\mathrm{Gr}_G^\mu}$.

Write $IC^{(2),\lambda\mu}$ for the IC sheaf of $\overline{\mathrm{Gr}^{(2),\lambda,\mu}_{G,\mathbb{A}^1}}$.

Extend this to IC sheaves on $\overline{\mathrm{Gr}^{(1),\lambda}_{G,C}} * \overline{\mathrm{Gr}^{(1),\mu}_{G,C}} \xrightarrow{m} \overline{\mathrm{Gr}^{(2),\lambda\mu}_{G,\mathbb{A}^1}}$ $\qquad \widetilde{IC}^{\lambda\mu}$, $IC^{(2),\lambda,\mu}$

(i.e., we're extending this IC sheaf using the inclusions of the $t\neq 0$ pieces of the above into the whole thing)

Facts: 1) m is small, so $m_! \widetilde{IC}^{\lambda\mu} = IC^{(2),\lambda,\mu}$

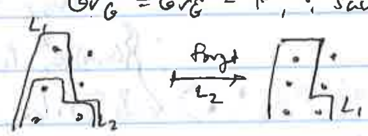
2) $\widetilde{IC}^{\lambda\mu}\big|_{t=0} \cong IC^{\lambda,\mu}$

Upshot: $IC^\lambda * IC^\mu = IC^{(2),\lambda\mu}\big|_{t=0}$

So we've found a symmetric way of describing the convolution product, by fusing fibers in the BD Grassmannian.

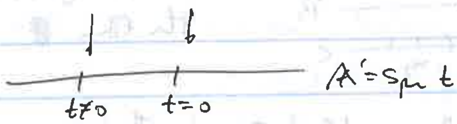
Ex $G = GL(2)$, $\lambda = (-1,0)$, $\mu = (0,-1)$. $\mathrm{Gr}_G^\lambda = \overline{\mathrm{Gr}_G^\lambda} = \mathbb{P}^1$, & same for μ.

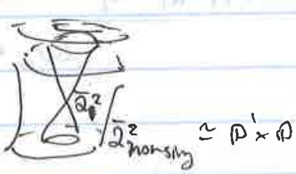
Convolution $\mathrm{Gr}_G^\lambda * \mathrm{Gr}_G^\mu = \mathbb{P}^1 * \mathbb{P}^1$
$\downarrow m$
$\mathrm{Gr}_G^{\lambda+\mu}$



This map looks like $\overline{T} * \mathbb{P}^1 \longrightarrow \overline{Q}_2$
$= \mathbb{P}^1 * \mathbb{P}^1$

BD picture $\overline{\mathrm{Gr}^{(2),\lambda\mu}_{G,\mathbb{A}^1}}$: $\mathbb{P}^1 \times \mathbb{P}^1 \rightsquigarrow \overline{Q}_2$ $\qquad x^2+y^2+z^2 = t^2$ degeneration of quadrics





So we have two ways of relating 2 copies of $\mathbb{P}^1$ to $\overline{Q}^2$:

- convolution involves a map $\mathbb{P}^1 * \mathbb{P}^1 \longrightarrow \overline{Q}^2$
- or we can study a degeneration of $\mathbb{P}^1 \times \mathbb{P}^1$ to $\overline{Q}^2$.

Now we're ready to finish the Tannakian dictionary which will prove the geometric Satake theorem.

We have seen that the symmetry discussed above allows us to equip $\mathrm{Perv}^{G(\mathcal{O})}(\mathrm{Gr}_G)$ with the structure of a $\otimes$-cat.

The fiber funct ($=$ faithful $\otimes$-funct to Vect) is given by global sections.

Tannakian Reconstruction: $\mathcal{C}$ $\otimes$-cat, $F: \mathcal{C} \longrightarrow \mathrm{Vect}$ $\otimes$-funct $\Rightarrow$ $\mathcal{C} = \mathrm{Rep}(H)$, where $H = \mathrm{Aut}^{\otimes}(F)$.

Ex $\mathcal{C} = \mathrm{Rep}(H)$, $F: \mathrm{Rep}(H) \to \mathrm{Vect}$ forgetful funct. Then $\varphi \in \mathrm{Aut}^{\otimes}(F)$ means for $V \in \mathrm{Rep}(H)$, we have $\varphi_V: F(V) \xrightarrow{\sim} F(V)$, with $\varphi_V \otimes \varphi_W \simeq \varphi_{V \otimes W}, \ldots$

So $\mathrm{Aut}^{\otimes}(F) = H$.

Applying this theory to $\mathrm{Perv}^{G(\mathcal{O})}(\mathrm{Gr}_G)$, we obtain an alg. group, which you can check is the combinatorially defined dual group.