4/20/2017

Geometric Satake: Convolution and BD Grassmannian.

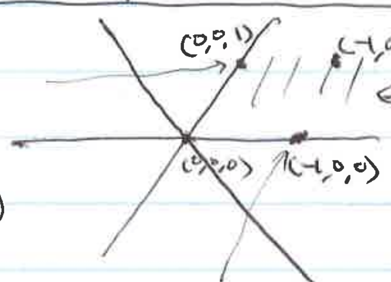
- $G = PGL(n) = GL(n)/GL(1)$. <u>then</u>:

- $Gr_G = \{ L \subset K^n \text{ lattices, identify } L \simeq t^k L, \forall k \in \mathbb{Z} \}$
  $(= Gr_{GL(n)}/GL(1))$

- $Fl_G = \{ L_0 \subset L_1 \subset \dots \subset L_n \text{ in } K^n, \text{ identify chains } L_0 \simeq t^k L_0, \forall k \in \mathbb{Z} \}$, $L_0 = t^{-1} L_n$
  $(= Fl_{GL(n)}/GL(1))$.

- $G^\vee = SL(n)$.

<u>Warmup Calculation</u>: $G = PGL(3)$, $G^\vee = SL(3)$.



h.w. of st. (standard rep)

$\Lambda_T^{dom.} = (\Lambda^\vee_{T^\vee})^{dom.}$

$\Lambda_T = \Lambda^\vee_{T^\vee}$ (Coweight lattice)

h.w. of $st^* \simeq \wedge^2 st$

$st \otimes st^* \simeq Ad \oplus Tr$ (trivial)

<u>Convolution</u>. $IC_{(-1,0,0)} * IC_{(0,0,1)} = \pi_* IC_Y$

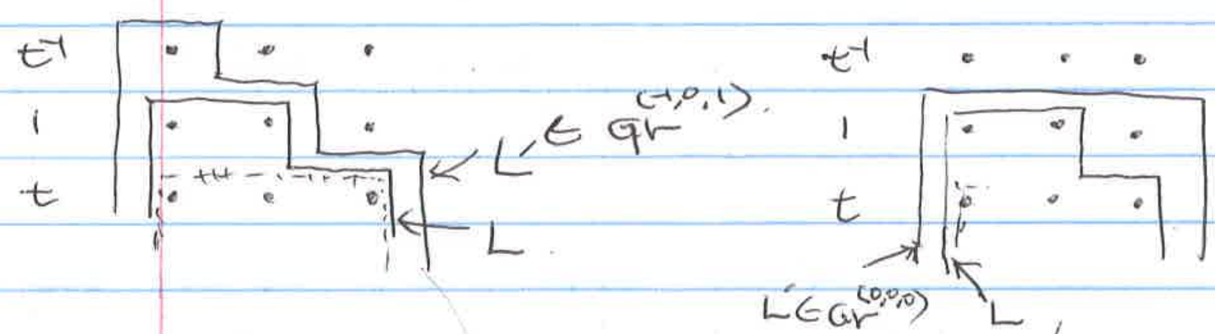
$IC_{(-1,0,0)} \leftrightarrow (L' \xleftarrow{(-1,0,0)} L)$, $IC_{(0,0,1)} \leftrightarrow L$, $(L', L) \longrightarrow L'$

$\pi : Y = \{ L \in Gr_G^{(0,0,1)}, L' \in Gr_G \text{ s.t. } L' \xleftarrow{(-1,0,0)} L \} \longrightarrow Gr_G$

2.

· rel. pos. $(L, \mathcal{O}^3) = (0,0,1)$

· rel. pos. $(L', L) = (-1,0,0)$.



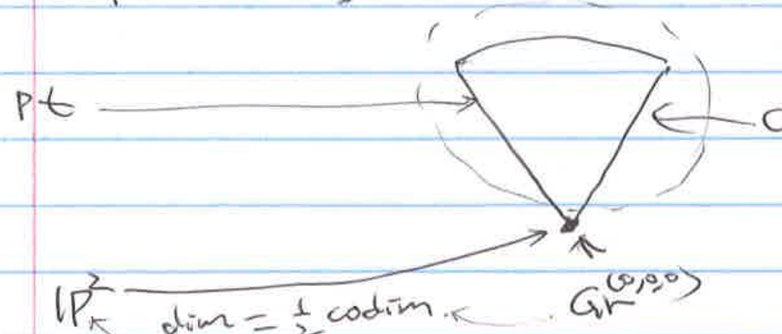
$\rightsquigarrow$: $Y = \underset{\ni L'}{\overset{Gr(1,3) =}{\mathbb{P}^2}}$-bdle over $\underset{\ni L}{\overset{Gr(2,3) =}{\mathbb{P}^2}}$

· Fibers of π:

· over $\underset{(Gr^{(-1,0,1)})}{(-1,0,1)}$ = pt

· over $(0,0,0)$ = $\mathbb{P}^2$.

$Y = \mathbb{P}^2 \tilde{\times} \mathbb{P}^2 \xrightarrow{\pi} \overline{Gr^{(-1,0,1)}}$

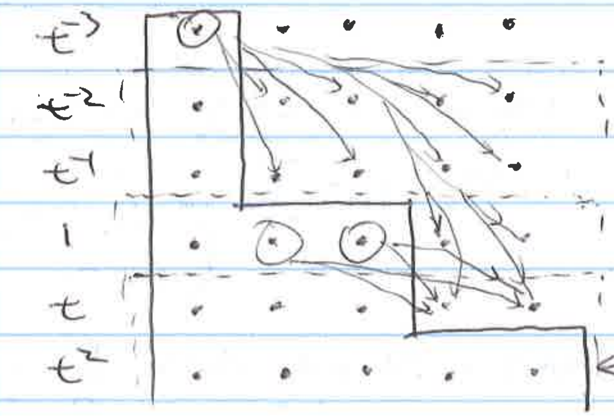


$\Rightarrow$ semismall.

· $Gr^{(-1,0,1)}$ = line bdle over G/B $\Rightarrow$ dim $= 4$.

In general, $G(\mathcal{O})$-orbits are vector bundles over (partial) flag varieties.

Example:



$\leftarrow L' \in Gr^{(-3,0,0,2,2)}$

$$\rightsquigarrow \quad Gr^{(-3,0,0,2,2)} \longrightarrow \{E_1 \subset E_3 \subset E_5 = \mathbb{C}^5\}$$

$$L' \longmapsto \left(t^3 \frac{L' \cap t^{-3}\mathcal{O}^5}{L' \cap t^{-2}\mathcal{O}^5} \hookrightarrow \frac{L' \cap \mathcal{O}^5}{L' \cap t\mathcal{O}^5} \hookrightarrow t^{-2} \frac{L' \cap t^2\mathcal{O}^5}{L' \cap t^3\mathcal{O}^5} \right)$$

is a rk 16 v-bundle.

Conclude: $IC_{(-1,0,0)} * IC_{(0,0,1)} = \pi_* IC_Y = IC_{(-1,0,1)} \oplus IC_{(0,0,0)}$.

Stalk of $\pi_* IC_Y$ at $Gr^{(0,0,0)}$:

$$H^*(\mathbb{P}^2) = \begin{array}{rl} 0 & \boxed{\mathbb{C}} \leftarrow IC_{(0,0,0)} \\ -2 & \mathbb{C} \\ -4 & \mathbb{C} \leftarrow IC_{(-1,0,1)}. \end{array}$$

$(\leftarrow Gr^{\mu})$

(· In general, stalk at μ of IC_λ is isom. to μ-weight space of V^λ.)

· why is convolution commutative?

Reinterpret $Gr_G = G(K)/G(\mathcal{O}) \simeq \{ G\text{-bdle } P \text{ on } D, \text{ with trivialization } P^0 \xrightarrow{\sim} P|_{D^*} \}$

$\xrightarrow{\sim} \{ G\text{-bdle } P \text{ on } C \text{ (smooth curve), with trivialization } P^0 \xrightarrow{\sim} P|_{C\setminus\{pt\}} \}$.

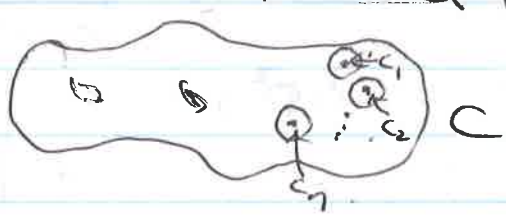
· $D^* = \operatorname{Spec} K$:  $\rightsquigarrow$ $G(K)$ = gluing data $P_1^0 \xrightarrow{\sim} P_2^0$.
(trivial G-bdles on D^*)

· $D = \operatorname{Spec} \mathcal{O}$.  $\rightsquigarrow$ $G(\mathcal{O})$ = automorphisms of P^0.
trivial G-bdle on D.

~~Observe~~ : ~~Gr~~

Defn : BD Grassmannian

$$Gr^{(n)}_{G,C} = \{ c=(c_1,\dots,c_n)\in C^n,\ P \text{ G-bdle on } C, \text{ with trivialization } P^0 \xrightarrow{\sim} P|_{C\setminus\cup\{c_i\}} \}$$

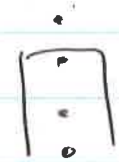


observe : $Gr^{(n)}_{G,C}\big|_{\overset{\circ}{C}{}^k} \simeq \prod^k Gr_G$

(where k pts distinct)

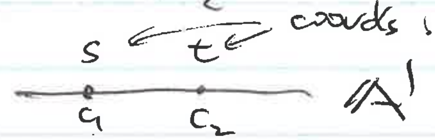
Lattice model for $G = GL(k)$, $C = \mathbb{A}^1$, $n = 2$

e.g. $G = GL(1)$, $Gr_{GL(1)} \simeq \mathbb{Z}$.

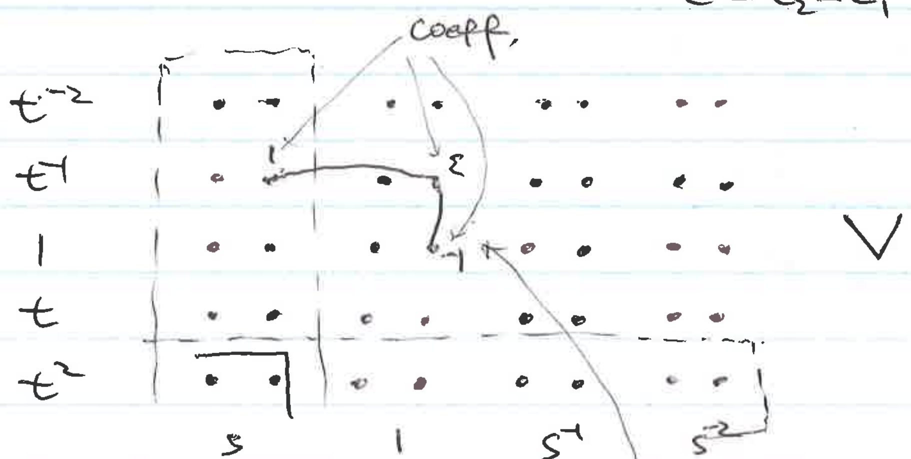


$\Rightarrow$ $Gr^{(n)}_{GL(1),C}$ = $\mathbb{Z}$-labelled divisors at n-pts

• consider $Gr^{(2)}_{GL(k),\mathbb{A}^1}$ over $\mathbb{A}^1_\varepsilon$:



$\varepsilon = c_2 - c_1$



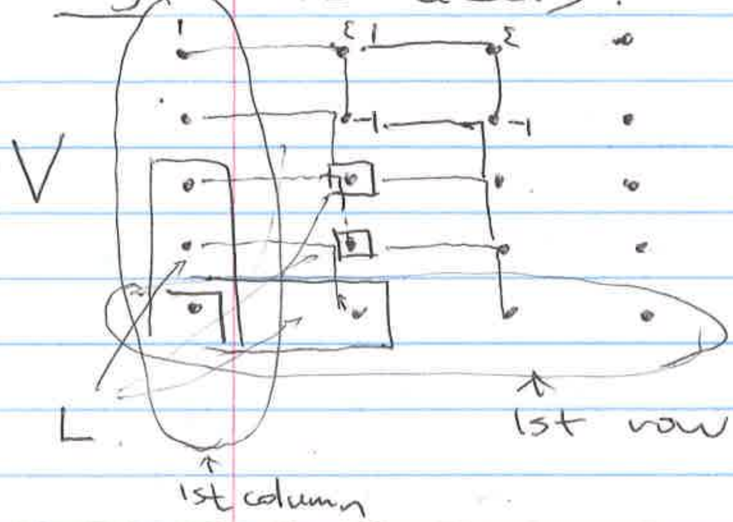
$L \subset V$ s.t.

1) L contains $t^N s^N \mathcal{O}^k$

2) $tL \subset L$, $sL \subset L$

3) L contains $(s - t + \varepsilon)\cdot v \in V$ $\forall v \in V$.

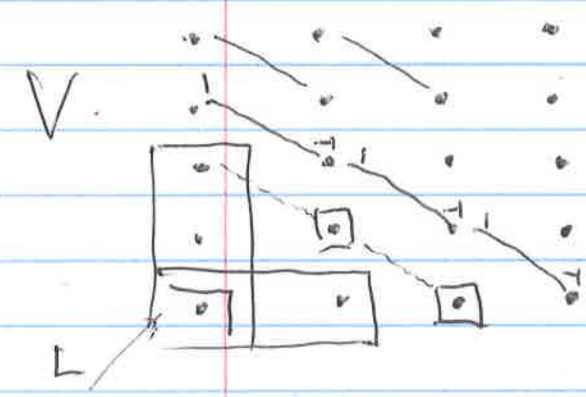
e.g. $G = GL(1)$.



$\underline{\varepsilon \neq 0}$: Any $L \subset V$ satisfying (1), (2), (3) is uniquely determined by:

$L \cap$ (1st. column), and

$L \cap$ (1st. row)



$\underline{\varepsilon = 0}$.