Apr. 18th, 2017

Last time: G reductive $\rightsquigarrow$ $Gr_G = G(K)/G(\mathcal{O})$.

$K = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$.

$Fl_G = G(K)/\mathcal{I}$.

$$\begin{array}{ccccc} \mathcal{I} & \subset & G(\mathcal{O}) & \subset & G(K). \\ \downarrow^{r} & & \downarrow^{\mathrm{ev}\ t=0} & & \\ B & \subset & G & & \end{array}$$

Ex: $G = GL_n$.

$Gr_G = \{ \text{Lattices } L \subset K^n \}$.

$Fl_G = \{ \text{Lattice chains}: tL_n = L_0 \subset L_1 \subset \cdots \subset L_n \}$.

· Bruhat / Cartan Decomposition

$$\mathcal{I} \backslash Fl_G = \mathcal{I} \backslash G(K)/\mathcal{I} = W^{aff} \quad \text{affine Weyl grp}$$

$$= W \ltimes \Lambda_T$$

$\llcorner$ coweight lattice.

$$G(\mathcal{O})\backslash Gr_G = G(\mathcal{O})\backslash G(K)/G(\mathcal{O})$$

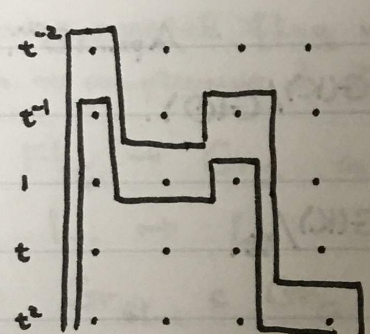
$$= W\backslash W^{aff}/W \simeq \Lambda_T/W$$

$$\simeq \Lambda_T^{dom}.$$

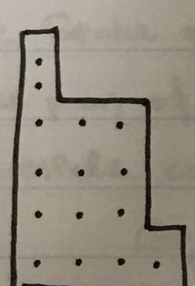
Ex: $G = GL_n$.

W^{aff} = perm. matrices with t^k's instead of 1's.

Lattice picture:



Skyline & corresponding flag $\in Fl_G$.



Dominant skyline. $\in Gr_G$.

· First Hint of Satake: $G(\mathcal{O})\backslash Gr_G \simeq \Lambda_T^{dom}$.

Recall: $T \subset B \subset G \rightsquigarrow T^\vee \subset B^\vee \subset G^\vee$

$$\Lambda_T = \Lambda^\vee_{T^\vee}$$

$$W = W^\vee$$

$$\text{irreps of } G^\vee \simeq (\Lambda^\vee_{T^\vee})^{dom}$$

Geometric Satake lifts bijection to equiv. of $\otimes$-cat's.

$$\mathrm{Perv}^{G(\mathcal{O})}(\mathrm{Gr}_G) \simeq \mathrm{Rep}_{\text{fin.dim}}(G^\vee)$$

· Rmk: full derived Satake equiv. of E_3-cat's.

$$\mathrm{Sh}_c^{G(\mathcal{O})}(\mathrm{Gr}_G) \simeq \mathrm{Coh}(\overbrace{\mathrm{Loc}_{G^\vee}(\mathbb{D})}^{\text{derived stack.}})$$

$$\mathbb{D} = D \sqcup_{D^\times} D.$$

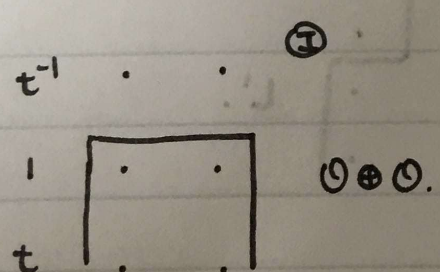
Note: $G(\mathcal{O})\backslash \mathrm{Gr}_G = \mathrm{Bun}_G(\mathbb{D})$.

Strategy: equip $\mathrm{Perv}^{G(\mathcal{O})}(\mathrm{Gr}_G)$ with <u>enough structure</u> $_{1\text{st}}$. to invoke <u>Tannakian formalism</u>. $_{\text{later}}$.

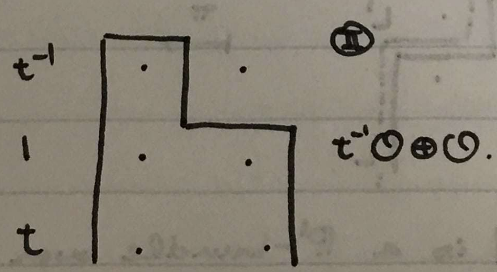
First structure: $\otimes$ product.

Miracle: convol'n product in $\mathrm{Sh}_c^{G(\mathcal{O})}(\mathrm{Gr}_G)$ preserves perv. sheaves.

Ex: $G = GL_2$.



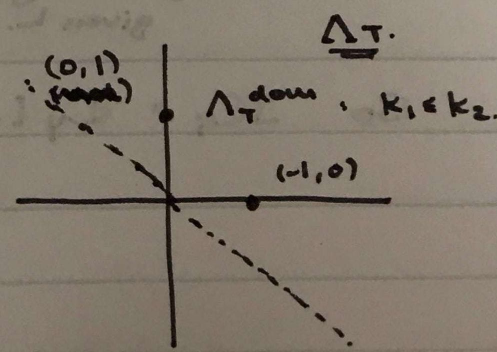
rel. dim 0.



rel. dim 1

$$T = \begin{pmatrix} t^{k_1} & 0 \\ 0 & t^{k_2} \end{pmatrix}.$$

$$\Lambda_T^{\text{dom}} \subset \Lambda_T = \mathbb{Z}^2.$$



Tannakian incarnation : $\overset{\text{I}}{\mathrm{Tr}}$, $\overset{\text{II}}{\mathrm{St}}$.

Notation: for $\lambda \in \Lambda^{dom}$.

$$IC_\lambda = \text{intersection cx. on } \overline{G(\mathcal{O})\cdot t^\lambda} \subset Gr_G.$$

Let's calculate:

$$IC_{(-1,0)} * IC_{(0,1)}$$

We know the answer (from rep theory)

$$St \otimes St^\vee \cong Ad = Triv \oplus Ad^{tr=0}.$$

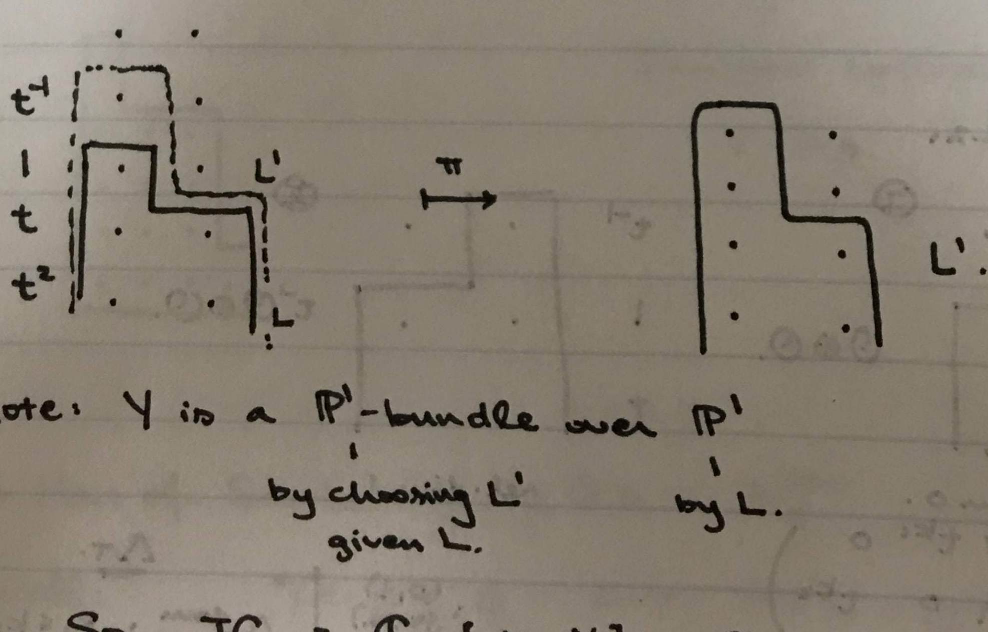
$$\Rightarrow IC_{(-1,0)} * IC_{(0,1)} = IC_{(0,0)} \oplus IC_{(-1,1)}.$$

$IC_{(-1,0)} * IC_{(0,1)} = \pi_* IC_Y$ where

$$Y = \{ L \in Gr^{(0,1)},\ L' \in Gr \text{ s.t. } rel.pos(L,L') = (-1,0) \}$$

$\pi : Y \to Gr$ by remembering L'.

Y parametrizing lattice pictures of the following type:



Note: Y is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$

by choosing L' given L. | by L.

So $IC_Y = \mathbb{C}_Y[\dim Y] = \mathbb{C}_Y[2]$.

Fibers:

- are nonempty over:

pt.	fiber
$Gr^{(-1,1)}$	pt.
$Gr^{(0,0)}$	$\mathbb{P}^1$.

Exercise: these are all the nonempty fibers.

Note: this is semismall.

$\Rightarrow \pi_* IC_Y = IC_{(-1,1)} \oplus IC_{(0,0)}$ no shifts!

Exercise: $Y = \mathbb{P}^1$-bundle given by $\mathcal{O}(-2)$.

$T^*\mathbb{P}^1$.