Lecture 2, Langlands Duality

01/19/2017

Summarizing where we left off, we work over $k = \mathbb{C}$ and $\mathfrak{g}$ being reductive means $\operatorname{ad}\mathfrak{g}$ is completely reducible, meaning $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g})$, where $[\mathfrak{g}, \mathfrak{g}]$ is semisimple. We recalled that semisimple means $\operatorname{rad}\mathfrak{g} = 0$, while reductive means $\operatorname{rad}\mathfrak{g} = Z(\mathfrak{g})$. In particular, the abelian Lie algebra k is never semisimple, but is reductive. Lastly, we said a group G being reductive meant that its finite dimensional representations are completely reducible, and this was equivalent to its unipotent radical (the $g \in \operatorname{rad}(G)$ such that $1 - g$ is nilpotent) being trivial. A finite dimensional representation of a reductive algebra $\mathfrak{g}$ might not be completely reducible; it is if its center acts as scalars.

Examples of Lie algebras

- $\mathfrak{p} = \{A : A$ is block upper-triangular, with the size of the blocks given by some partition of the dimension$\}$. Borel subalgebra is a special case. This is not reductive.
- $\mathfrak{g} = \{A : A$ is block diagonal$\}$ (with sizes given again arbitrary). This is reductive, but has a center (one dimension for each block), so it isn't semisimple.
- $\mathfrak{g}_{ss} = [\mathfrak{g}, \mathfrak{g}]$: block diagonal matrices where each block is traceless. This is semisimple, and is simple when there is just one block, i.e. $\mathfrak{g}_{ss} = \mathfrak{sl}(n)$.

Examples of Lie groups There was one additional component in the hierarchy of Lie groups vs. algebras: affine groups lie between complex Lie groups and semisimple Lie groups.

- $G = S^1$ is not complex.
- $G = E/\mathbb{C}$, a complex elliptic curve, is compact and non-affine.
- $\mathfrak{G}_a = \mathfrak{A}^1$ is affine but not reductive (even though its algebra is reductive).
- $\mathfrak{G}_m = GL(1)$ is reductive (like its algebra), but not semisimple.
- $G = SO(4)$ is semisimple but not simple ($SO(4) \cong SO(2) \times SO(2)/\{\pm 1\}$)

Classification of simple Lie algebras Subscripts indicate rank, and inequalities for n are chosen so there are no repetitions.

- $A_n = \mathfrak{sl}(n+1); n \geq 1$
- $B_n = \mathfrak{so}(2n+1); n \geq 2$

- $C_n = \mathfrak{sp}(2n); n \geq 3$
- $D_n = \mathfrak{so}(2n); n \geq 4$
- E_6, E_7, E_8
- F_4
- G_2

Exercise: Construct some of the exceptional algebras, or their groups.

Classification of simple Lie groups Every group has a universal cover, and all simple Lie groups have finite centers. Hence each of the algebras above corresponds to finitely many non-isomorphic Lie groups.

- A_n: The group $SL(n+1)$ is simply-connected (we're working over $\mathbb{C}$, not $\mathbb{R}$), and its center consists of scalars, the $n+1^{\text{th}}$ roots of unity. Lie groups of type A_n are thus classified by subgroups of the cyclic group C_{n+1} of size $n+1$; there are as many of these as there are divisors of $n+1$. At the opposite side of $SL(n)$, quotienting out the whole center gives $PGL(n+1) = SL(n+1)/Z(SL(n+1)) = GL(n+1)/GL(1)$, whose fundamental group is C_{n+1}, and is actually simple as a (not-Lie) group.
- B_n: All orthogonal groups have fundamental group $\pi_1(SO(m)) = \mathbb{Z}/2$. Odd dimensional ones have no center, so there are two groups of type B_n: $SO(2n+1)$ and its universal cover, the spin group $Spin(2n+1)$.
- C_n: The symplectic group $Sp(2n)$ is simply connected, like with the A_n series. The center of $Sp(2n)$ is plus/minus the identity, so $Sp(2n)/\{\pm 1\}$ is the only other group of type C_n.
- D_n: There is $SO(2n)$, $Spin(2n)$, and as noted above, $SO(2n)/\{\pm 1\}$. Things get even slightly worse for this case though: when n is even, the center of $Spin(2n)$ is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$; when n is odd, it is $\mathbb{Z}/4$.
- So now we "know" the classification of reductive groups: quotients by discrete subgroups of finite products of groups in the above list with abelian groups.

Langlands dual algebras and groups

On Lie algebras, the dual preserves direct sums and abelian components. On simple algebras, it switches B_n and C_n and preserves every other type, and on abelian algebras, $\mathfrak{z}^\vee = \mathfrak{z}^*$ (Langlands dual is vector space dual). So on semisimple Lie algebras, the Langlands dual is the Dynkin dual (the Lie algebra with dual Dynkin diagram). These facts define the Langlands dual for any reductive complex Lie algebra, but don't explain where the concept comes from.

On the group level, for G simple, it "switches" the fundamental group and center. So

$$SL(n)^\vee = PGL(n), SO(2n+1)^\vee = Sp(2n), Spin(2n)^\vee = SO(2n)/\{\pm 1\}, SO(2n)^\vee = SO(2n).$$

Somehow this is supposed to act like a nonabelian Fourier transform. One theme of this course will be that representation theory and harmonic analysis on G corresponds to algebraic geometry on $G^\vee$.

Definition of the Langlands dual for tori Exercise: A group T is reductive and abelian implies $T \cong GL(1)^n = (\mathbb{C}^*)^n$. This is what we call a torus (it deformation retracts onto the torus $(S^1)^n$).

For a torus T, we defined the *character / weight lattice* $X^*(T) = \Lambda_T^\vee$ as the set of homomorphisms (as abelian groups) $T \to GL(1)$. For each factor of $\mathbb{C}^*$ in T, this is the choice of some m-sheeted covering map for some integer m ($m = 0$ corresponds to the constant map). So $X^*(T) \cong \mathbb{Z}^n$.

We defined the *cocharacter / coweight lattice* $X_*(T) = \Lambda_T$ of T as $\mathrm{Hom}_{\mathbf{Ab}}(\mathbf{G}_m, T)$, and this is again isomorphic to $\mathbb{Z}^n$. The two are dual lattices.

Exercise: Show $T = \operatorname{Spec} \mathbb{C}[\Lambda_T^\vee] = \Lambda_T \otimes_{\mathbb{Z}} \mathbf{G}_m$.

With that we define the Langlands dual $T^\vee$ of T to be $\operatorname{Spec} \mathbb{C}[\Lambda_T] = \Lambda_T^\vee \otimes_{\mathbb{Z}} \mathbf{G}_m$.

Exercise: $\Lambda_{T^\vee}^\vee = \Lambda_T, \Lambda_{T^\vee} = \Lambda_T^\vee$.

In contrast to what we'll see later, everything defined so far here for tori was clearly functorial. Since the character lattice is the set of possible characters of the irreducible representations of T (they're all 1-D since T is abelian), and since T is reductive, this means that the category of f.d. representations of T is graded by $\Lambda_T^\vee$. The statement of the *Satake correspondence* is then that this category is equivalent, as a tensor category, to the category of $\Lambda_T^\vee$-graded vector spaces. On the representation side, the tensor product is the tensor product of representations; this gives a convolution product on the vector space side, which defines the tensor category structure there.