6 April 2017

<u>Next topic:</u> Geometric Satake equivalence: description of "Spherical Hecke category"
(For comparison: our prior focus has been "finite Hecke category")

~~[illegible]~~	Arithmetic	Geometric
Local field (function on $D^\times$)	$\mathbb{Q}_p$ p-adic numbers (or extensions)	$\mathbb{F}((t))$ Laurent series (or extensions)
	↖ complete $\mathbb{Z}$ at a prime, then pass to field of fractions	← Geometric analogue of this
(function on D) Ring of integers (coordinate)	$\mathbb{Z}_p$ p-adic integers	$\mathbb{F}[[t]]$ power series
Uniformizer	$\overset{\Psi}{p}$	$\overset{\Psi}{t}$
(function on point) Residue field	$\mathbb{F}_p$ (or finite extension)	$\mathbb{F} = \mathbb{F}_p, \mathbb{C}, \ldots$ These are the two traditional choices
Finite-dim. group	$G(\mathbb{F}_p)$	$G(\mathbb{F})$ (from now on, we'll take $\mathbb{F}=\mathbb{C}$)
Flag vars, or Parabolic subgroups	$X_p = G/P$, $P \subset G$ ⟷ vertices of Dynkin diagram, e.g.	
	our favorite: Borel = min parabolic $B \subset G$ ⟷ no vertices of diagram ⟵ X_B full flags	
Group over local field	$G(\mathbb{Q}_p)$ p-adic group	$G(\mathbb{C}((t)))$ loop groups, affine KM groups ← call this $\mathcal{G}$.
affine Flag vars / parahoric subgroups	$X_{\mathcal{P}} = \mathcal{G}/\mathcal{P}$, $\mathcal{P} \subset \mathcal{G}$ ⟷ vertices of <u>affine</u> DD, e.g.	
	analogue of Borel: Iwahori $X_{\mathcal{I}} = \mathcal{G}/\mathcal{I}$, $\mathcal{I}$ = min. parahoric ⟷ no vertices of diagram	
Linearization	(smooth) functions (with compact support...) [this is a vector space]	(constructible) sheaves (w/ cpct support...) [this is a category]

What are affine flag varieties? What are parahoric subgroups?
(They are <u>not</u> $P(\mathbb{C}((t)))$.)

<u>Answer for $GL(n)$:</u> Set $K = \mathbb{C}((t))$. Then $GL(n, K)$ = $n\times n$ invertible matrix with K-entries.
We'd like to invent a flag variety for this. (Parahorics will be stabilizers)
$GL(n,K) \curvearrowright K^n$ linearly. A topological basis for K^n is $\{t^i e_k\}_{k=1,\ldots,n}^{i\in\mathbb{Z}}$.

$$\begin{array}{c|cccc}
\vdots & & & & \\
t^{-2} & t^{-2}e_1 & t^{-2}e_2 & \cdots & t^{-2}e_n \\
t^{-1} & t^{-1}e_1 & & \cdots & t^{-1}e_n \\
1 & e_1 & & \cdots & e_n \\
t & te_1 & & \cdots & te_n. \\
\vdots & & & &
\end{array}$$

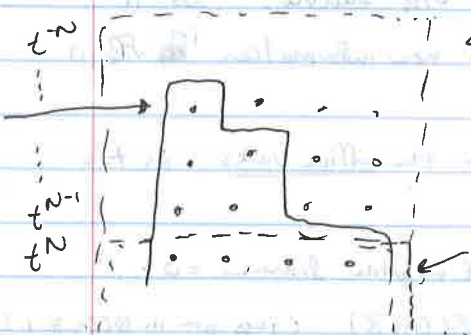
To save writing, let's just write dots for basis vectors:

Set $\mathcal{O} = \mathbb{C}[[t]]$.

Defn: A <u>lattice</u> is a subspace $L \subset K^n$ such that: 1) $L \supset t^N \mathcal{O}^n$ for $N >> 0$

2) $L \subset t^{-N}$ for $N >> 0$.

3) L is an $\mathcal{O}$-module ($\Rightarrow$ if $t^j e_k \in L$, then $t^{j+1} e_k \in L$)

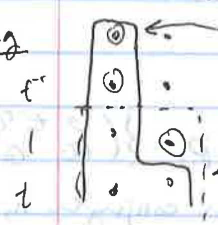


[<u>Warning</u>: Not all subspaces are coordinate subspaces, but those are the ones that are easy to draw]

<u>Note:</u> L doesn't have a (finite) dimension or codimension, but it does have a well-defined <u>relative dimension</u> with respect to $\mathcal{O}^n$,

defined as $\text{rel dim } L := \dim(L/L \cap \mathcal{O}^n) - \dim(\mathcal{O}^n / L \cap \mathcal{O}^n) \in \mathbb{Z}$.

<u>e.g.</u>

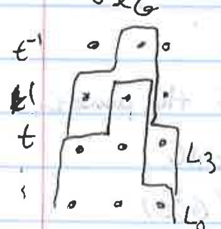


rel dim $= 2 - 1 = 1$.

<u>Affine flag</u> <u>variety</u> (for $G = GL(n)$)

$Fl_G = X_I = \{L_0 \subset L_1 \subset \cdots \subset L_n \text{ lattices} \mid \dim L_i / L_{i-1} = 1,\ tL_n = L_0\}$

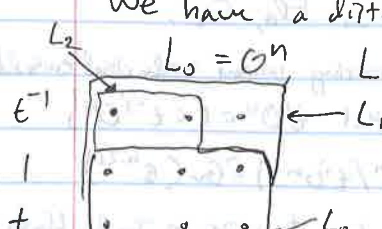
<u>Ex</u>



<u>Note</u>: Once you've chosen L_0, you have a (finite-dim) flag var's worth of choices! $L_1 \subset \cdots L_{n-1}$ give full flag in $L_n / L_0 \simeq \mathbb{C}^n$.

We have a distinguished point in Fl_G given by

$L_0 = \mathcal{O}^n \quad L_k = \mathcal{O}^n + t^{-1}\langle e_1, \ldots, e_k \rangle, \quad L_n = t^{-1}\mathcal{O}^n$



let's call this point $L_\cdot^{base} \in Fl_G$

<u>Lemma</u> $G(K)$ acts transitively on Fl_G.

$I := \text{Stab}(L_\cdot^{base}) = \{g \in G(\mathcal{O}),\ g(0) \in B\}$. $\qquad Fl_G = G(K)/I$.

Now the partial flag varieties come from deciding to forget some of the L_i.

We have a favorite partial flag variety, the affine Grassmannian

$$Gr_G = G(\mathcal{K})/G(\mathcal{O}) = \{\text{lattices } L \subset \mathcal{O}^n\}$$

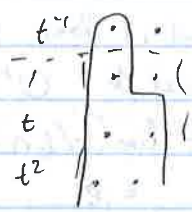
We saw before that $Fl_G \longrightarrow Gr_G$ is a G/B bundle. So if you're already a master of G/B, all the new information in Fl_G is contained in Gr_G.

Gr_G corresponds to choosing everything except the affine vertex in the affine Dynkin diagram.

Rmk: $Gr_{SL(n)} \subset Gr_{GL(n)}$ is cut out as lattices of relative dimension $=0$.

the other max parahorics are not conj. in $SL(n, \mathcal{K})$ (they are in $GL(n,\mathcal{K})$)

e.g.



rel dim $=0$ rel dim $=1$. Stab$(L_1) = P_1 := \left\{\begin{pmatrix}\mathcal{O} & t^{-1}\mathcal{O}\\ t\mathcal{O} & \mathcal{O}\end{pmatrix}\right\}$.

↖ in here, have favorite L_0. Stab$(L_0) = \left\{\begin{pmatrix}\mathcal{O} & \mathcal{O}\\ \mathcal{O} & \mathcal{O}\end{pmatrix}\right\} = P_0$

These two are conjugate in $GL(2,\mathcal{K})$ but not $SL(2,\mathcal{K})$.

The Geometric Satake equivalence will be a description of the general part of the Hecke category of $G(\mathcal{O})$-equiv constructible complexes (with compact support) on $Gr_G = G(\mathcal{K})/G(\mathcal{O})$

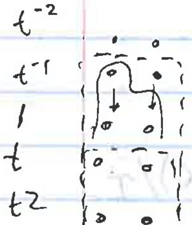
Structure of affine flag vars.

We'll focus on Fl_G, Gr_G.

These are ind-projective varieties (in fact, increasing unions under closed embeddings).

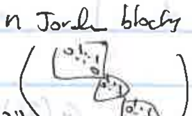
$Gr_G = \bigcup_{i\in\mathbb{N}} Gr_G^i$, $\quad Gr_G^i := \{\text{lattices } L \subset \mathcal{K}^n \text{ such that } t^i\mathcal{O}^n \subset L \subset t^{-i}\mathcal{O}^n\}$

$Gr_G^0 = \{\mathcal{O}^n\} = pt.$ $\quad Gr_G^i \hookrightarrow Gr(t^{-i}\mathcal{O}^n/t^i\mathcal{O}^n) \cong Gr(\mathbb{C}^{n\cdot 2i})$.



Gr_G^1

$\mathbb{C}^{n\cdot 2i}$ has a nilp. operator t with n Jordan blocks of size $2i \cdot 2i$, and



$Gr_G^i \xrightarrow{\sim} \{E \mid tE \subset E\} \hookrightarrow Gr(\mathbb{C}^{n\cdot 2i})$.

Exercise: this ↗ is a projective subvariety of $Gr(\mathbb{C}^{n\cdot 2i})$.