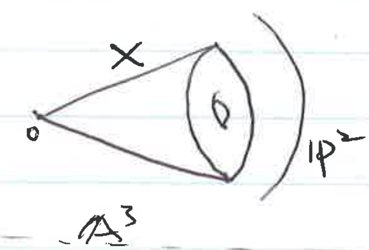
4/4/2017

Bott-Samelson resolutions

Warmup. $C \hookrightarrow \mathbb{P}^2$ smooth compact, plane curve.

- $X = \mathrm{cone}(C) \subset \mathbb{A}^3$ sing. surface.

$$\begin{array}{c} X \\ \uparrow \pi \\ \tilde{X} \end{array} = Bl_0(X).$$



Calculate. 1) IC_X

2) $\pi_* \mathbb{C}_{\tilde{X}}$

1) only singular pt of X is 0, so:

- $IC_X|_{X \setminus \{0\}} \simeq \mathbb{C}_{X\setminus\{0\}}[2]$. $(\dim_{\mathbb{C}} X = 2)$

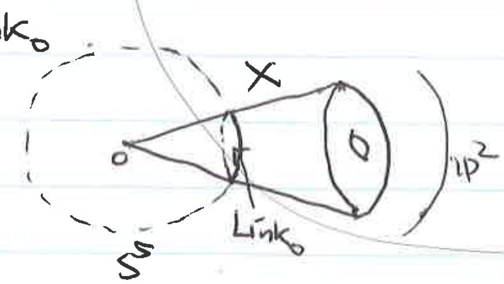
$$\begin{array}{rl} 2 & C^4 \\ & \uparrow \\ 1 & C^3 \\ & \uparrow \\ 0 & C^2 \\ & \uparrow \\ -1 & C^1 \\ & \uparrow \\ -2 & C^0 \quad (\leftarrow \mathbb{C}_{X\setminus\{0\}}[2]) \end{array}$$

- $IC_X|_{\{0\}} = \Gamma(X; IC_X)$
since X contracts to 0.

Recall. $IC_X|_{\{0\}} = \tau_{\leq -1}(C^*(\underset{\substack{\uparrow \\ \text{3-mfd.}}}{\mathrm{Link}_0})[2])$

- Calculates H^0, H^1 of Link_0

$\mathrm{Link}_0 = S^5 \cap X$..



$\rightsquigarrow$ fibration. $S^1 \longrightarrow \mathrm{Link}_0$

$$\begin{array}{c} \downarrow \\ C \end{array}$$

// circle bundle of $\mathcal{O}(-1)$ on C.

$(= \mathcal{O}_{\mathbb{P}^2}(-1)|_C)$.

- In particular, S^1-bundle is <u>oriented</u>.

$\rightsquigarrow$ <u>spectral sequence</u> for $H^*(\mathrm{Link}_0)$:

E_2 page:

$$\begin{array}{c|ccc} 1 & \mathbb{C} & \mathbb{C}^{2g} & \mathbb{C} \\ 0 & \mathbb{C} & \mathbb{C}^{2g} & \mathbb{C} \\ \hline & 0 & 1 & 2 \end{array}$$

$d_2 \neq 0$ since $c_1(\text{line bdle}) \neq 0$.

$$\Rightarrow H^*(\mathrm{Link}_0) = \begin{array}{cc} 3 & \mathbb{C} \\ 2 & \mathbb{C}^{2g} \\ 1 & \mathbb{C}^{2g} \\ 0 & \mathbb{C} \end{array}$$

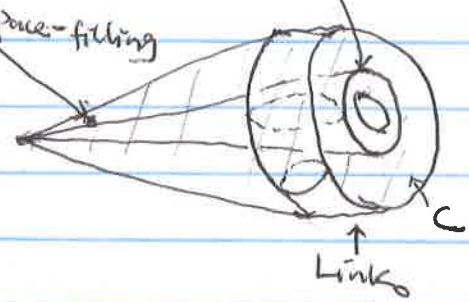
- <u>Conclude</u>:

$$IC_X|_{\{0\}} = \begin{array}{cc} 2 & 0 \\ 1 & 0 \\ 0 & 0 \\ -1 & \mathbb{C}^{2g} \\ -2 & \mathbb{C} \end{array}$$

cone(T^2)

space-filling

<u>Cartoon</u>.



2) $\pi : \tilde{X} \to X$, $\tilde{X}$: smooth, ~~π is semismall~~

$$\Rightarrow \quad \pi_* \mathbb{C}_{\tilde{X}}[2] \underset{\substack{\text{Decomp.}\\ \text{Thm}}}{=\!=} \bigoplus_{(X_\lambda, L_\lambda, d_\lambda)} IC_{X_\lambda}(L_\lambda)[d_\lambda]$$

• π is <u>semismall</u>: • $\dim(\underbrace{\text{fiber over } X\setminus\{0\}}_{=0}) \le \frac{1}{2}\underbrace{\operatorname{codim}(X\setminus\{0\})}_{=0}$

• $\dim(\underbrace{\text{fiber over } \{0\}}_{=1}) \le \frac{1}{2}\underbrace{\operatorname{codim}(\{0\})}_{=1}$

<u>upshot</u>: π is semismall $\Rightarrow d_\lambda = 0$.

<u>over $X\setminus\{0\}$</u>: π is isom, so IC_X occurs once in summand.

• <u>At $\{0\}$</u>: we need stalk isom.

$$\pi_* \mathbb{C}_{\tilde{X}}[2]\big|_{\{0\}} \simeq IC_X\big|_{\{0\}} \oplus IC_{\{0\}}^{\oplus k}\big|_{\{0\}}$$

$IC_{\{0\}}^{\oplus k}\big|_{\{0\}} = \mathbb{C}_{\{0\}}^{\oplus k}\big|_{\{0\}} = \mathbb{C}^{\oplus k} \Rightarrow \underline{k=1}$.

	$\pi_* \mathbb{C}_{\tilde{X}}[2]\big\vert_{\{0\}}$		$IC_X\big\vert_{\{0\}}$		$IC_{\{0\}}^{\oplus k}\big\vert_{\{0\}}$
$H^*(C)[2]$: 0	$\mathbb{C}$	0	0	0	$\mathbb{C}$
-1	$\mathbb{C}^{2g}$	-1	$\mathbb{C}^{2g}$	-1	0
-2	$\mathbb{C}$	-2	$\mathbb{C}$	-2	0

"π semismall, # of IC_{X_λ} in $\pi_* IC_{\tilde{X}}$ = # of conn. comp. of the fiber over X_λ".

4.

<u>Back to Hecke Category</u> $P_c^B(G/B)$ (B-equiv. constr. derived cat.)
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| $
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad X$

<u>Work with IC-basis</u>:

- $w \in W \rightsquigarrow$ Schubert cell $X_w = G_w/B$, $G_w = BwB$.

- IC_w denotes IC of $\overline{X_w}$. (IC-basis for $P_c^B(X)$)
  "Any obj. is a finite complex of IC_w's".

- <u>Key pt</u>. Decomposition thm
  $\Rightarrow$ <u>convolution</u>. $IC_{w_1} * IC_{w_2} = \bigoplus_i IC_{w_i}[d_i]$.

<u>Recall</u>. <u>Convolution</u>.

- $$B\backslash G/B \times B\backslash G/B \xleftarrow{p} B\backslash G \overset{B}{\times} G/B \xrightarrow{q} B\backslash G/B.$$
  smooth, fibers $\simeq B$ $\quad\quad$ proper, fibers $\simeq G/B$.

- $\mathcal{F}_1 * \mathcal{F}_2 := q_! \, p^* (\mathcal{F}_1 \boxtimes \mathcal{F}_2)$.

$$\rightsquigarrow B\backslash \overline{G_{w_1}}/B \times B\backslash \overline{G_{w_2}}/B \xleftarrow{p} B\backslash \overline{G_{w_1}} \overset{B}{\times} \overline{G_{w_2}}/B \xrightarrow{q} B\backslash G/B.$$

$$IC_{w_1} \boxtimes IC_{w_2} \xrightarrow{p^*} IC \text{ (up to shift)}$$
(since p is smooth)

· upshot. $\pi = G_{w_1} \overset{B}{\times} G_{w_2}/B \longrightarrow G/B$

Need to calculate. $\pi_* IC = \oplus IC's$.

Bott–Samelson resolutions.

· W. gen. by simple reflections s_i's.

· suppose $w = s_{i_1} \cdots s_{i_r}$, reduced expression.

$\Rightarrow \ell(w) = r$.

· Let $\tilde{w} = (s_{i_1}, s_{i_2}, \dots, s_{i_r})$. $\rightsquigarrow$

$$\left((E_r^\bullet \overset{s_{i_1}}{\leftarrow} E_{r-1}^\bullet \overset{s_{i_2}}{\leftarrow} \cdots E_1^\bullet \overset{s_{i_r}}{\leftarrow} E_0^\bullet) \overset{E^\bullet_{st}}{\longmapsto} E_r^\bullet \right)$$

$$\overline{X_{\tilde{w}}} = \overline{G_{s_{i_1}}} \overset{B}{\times} \overline{G_{s_{i_2}}} \overset{B}{\times} \cdots \overset{B}{\times} \overline{G_{s_{i_r}}}/B \xrightarrow{\pi} \overline{G_w/B} = \overline{X_w}.$$

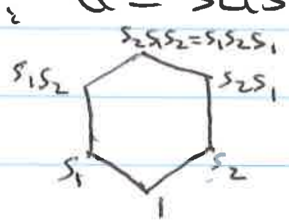
(iterated $\mathbb{P}^1$-bundles $\Rightarrow$ smooth).

"Bott–Samelson resolution".

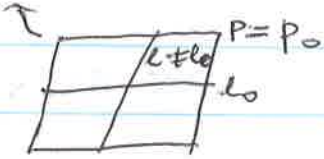
· π is proper, isom. over G_w/B.

$\Rightarrow IC_{s_{i_1}} * \cdots * IC_{s_{i_r}} = IC_w \oplus IC's[?]$
lower

Example. $G = SL(3)$, $s_1 = (12)$, $s_2 = (23)$.

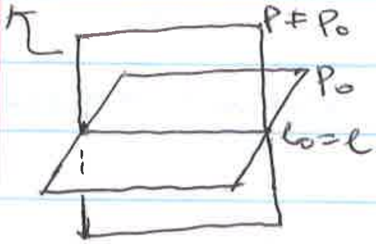
$\rightsquigarrow$ 

$\Rightarrow \cdot\ X_{s_1} = \{\ell \subset P \mid \ell \neq \ell_0,\ P = P_0\} \simeq \mathbb{A}^1.$



$\Rightarrow \cdot\ \overline{X_{s_1}} = \{\ell \subset P \mid P = P_0\} \simeq \mathbb{P}^1.$

$\cdot\ X_{s_2} = \{\ell \subset P \mid \ell = \ell_0,\ P \neq P_0\} \simeq \mathbb{A}^1.$

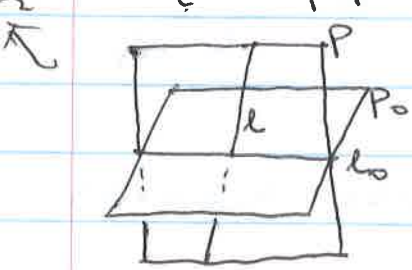


$\Rightarrow\ \overline{X_{s_2}} = \{\ell \subset P \mid \ell = \ell_0\} \simeq \mathbb{P}^1.$

$\cdot\ \underline{w = s_1 s_2}:$

$$\begin{tikzcd}
\pi : \overline{G_{s_1}} \overset{B}{\times} \overline{G_{s_2}}/B \arrow[r] \arrow[d, equal] & \overline{G_{s_1 s_2}}/B \arrow[d, equal] \\
\mathbb{P}^1\text{-bdle over } \mathbb{P}^1 \arrow[r] & \overline{X_{s_1 s_2}} \\
\left((\ell_1, P_1) \xleftarrow{s_1} (\ell_0, P_1) \xleftarrow{s_2} (\ell_0, P_0)\right) \arrow[r] & (\ell_1, P_1)
\end{tikzcd}$$

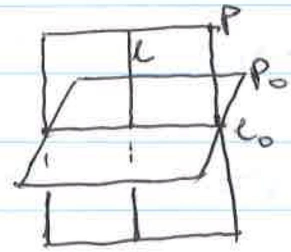
$\cdot\ \overline{X_{s_1 s_2}} = \{\ell \subset P \mid \ell_0 \subset P\}$



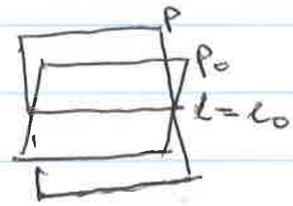
$\overline{X_{s_1 s_2}} = \bigcup_{w \le s_1 s_2} X_w \rightsquigarrow$

<u>Fibers of π</u>:

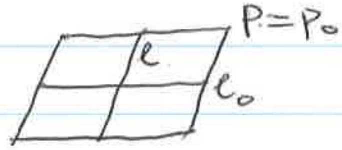
$s_1 s_2$: pt. $\leftsquigarrow$
(over $X_{s_1 s_2}$)



<u>s_2</u> : pt $\leftsquigarrow$
(over X_{s_2})



<u>s_1</u> : pt $\leftsquigarrow$



<u>1</u> : pt $\leftsquigarrow$

$\Rightarrow$ <u>π is Bom</u>.

<u>$w = s_1 s_2 s_1$</u>. $\left(\tilde{w} = (s_1, s_2, s_1) \right)$.

$\pi : \overline{G_{s_1}} \overset{B}{\times} \overline{G_{s_2}} \overset{B}{\times} \overline{G_{s_1}} / B \longrightarrow \overline{G_{s_1 s_2 s_1}} / B$

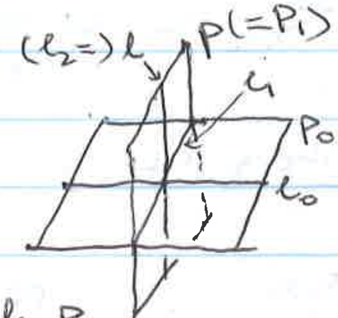
$\mathbb{P}^1$-bdle over $\mathbb{P}^1$-bdle over $\mathbb{P}^1 \longrightarrow \overline{X_{s_1 s_2 s_1}} = X$

$= \{ \ell \subset P \}$

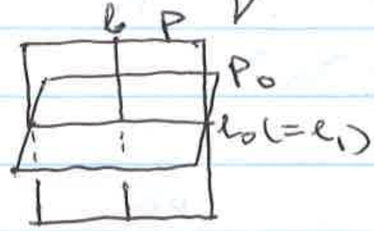
$\left((\ell_2, P_1) \overset{s_1}{\leftsquigarrow} (\ell_1, P_1) \overset{s_2}{\leftsquigarrow} (\ell_1, P_0) \overset{s_1}{\leftsquigarrow} (\ell_0, P_0) \right) \longrightarrow (\ell_2, P_1)$

Fibers of π :

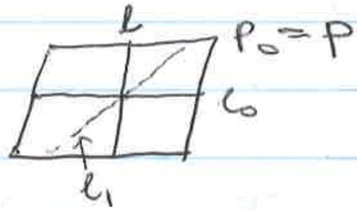
$s_1 s_2 s_1 = s_2 s_1 s_2$: pt $\leftsquigarrow$
(over $X_{s_1 s_2 s_1}$)



$s_1 s_2$: pt $\leftsquigarrow$

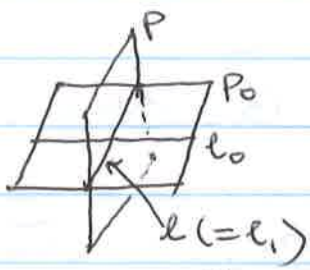


s_1 : $\mathbb{P}^1$ $\leftsquigarrow$

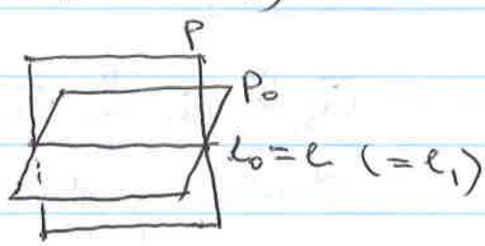


1 : $\mathbb{P}^1$ $\leftsquigarrow$

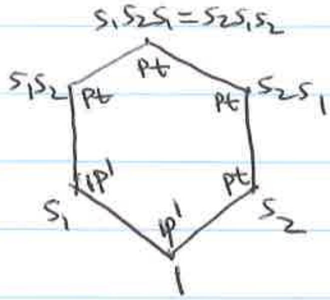
$s_2 s_1$: pt $\leftsquigarrow$



s_2 : pt. $\leftsquigarrow$



$\hookrightarrow$



Conclude:

$$\pi_* \mathbb{C}_{X_{\tilde{w}}}[3] = IC_X \oplus IC_{s_1}.$$

for $\tilde{w} = (s_1, s_2, s_1)$.