3/23/2017

Examples of IC complexes

X Var.
$\cup$
X_λ strata, L_λ local system on X_λ

$\rightsquigarrow IC_{X_\lambda}(L_\lambda) \in Perv(X)$.

---

1) Curves.

$j : X^\circ \hookrightarrow X$ smooth locus.

Construct IC_{X°.

X 

$\mathcal{L}^\circ = \mathbb{C}_{X^\circ} \rightsquigarrow IC_{X^\circ} \in Perv(X)$.

Step 1. Start with $\mathbb{C}_{X^\circ}[1]$.

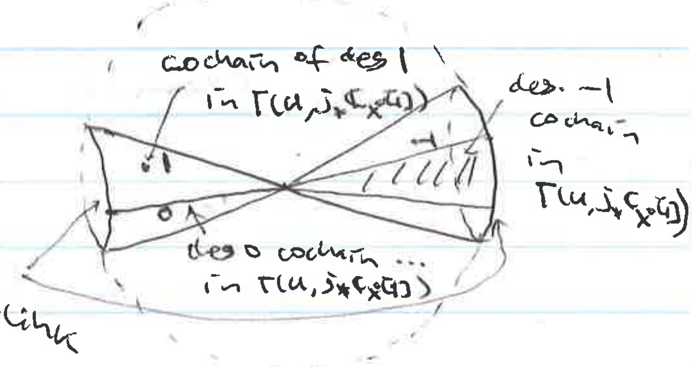
Step 2. pushforward $j_* \mathbb{C}_{X^\circ}[1]$
and truncate $\tau^{\le -1} j_* \mathbb{C}_{X^\circ}[1] =: IC_{X^\circ}$
$(X_\alpha = X^\circ,\ X_\beta = X \setminus X^\circ \hookrightarrow \overline{X_\alpha}.\ \tau^{\le -\dim X_\beta - 1} = \tau^{\le -1})$.

- what is this?
- over X°, recover $IC_{X^\circ}|_{X^\circ} = \mathbb{C}_{X^\circ}[1]$.

2.

• At a sing. pt:

stalk of $j_* \mathbb{C}_{X^\circ}[1]$



$= \Gamma(U; j_*\mathbb{C}_{X^\circ}[1])$

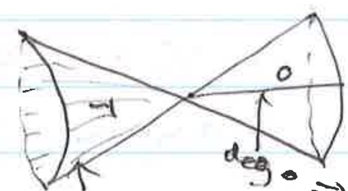
$= C^\cdot(U \cap X^\circ)[1]$

Calculate the coh. homology of link

U open $\hookrightarrow X$ near a sing. pt

then truncate stalk:

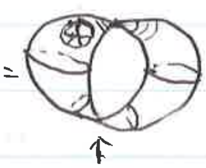
$\tau^{\le -1}(\Gamma(U, j_*\mathbb{C}_{X^\circ}[1]))$ calculates H^0 of link.



deg. -1 cochain, in $\tau^{\le -1}\Gamma(U, j_*\mathbb{C}_{X^\circ}[1])$, whose coboundary vanishes

	$\Gamma(U, j_*\mathbb{C}_{X^\circ}[1])$	$\tau^{\le -1}\Gamma(U, j_*\mathbb{C}_{X^\circ}[1])$
1	$C^2_{X^\circ}$	0
0	$C^1_{X^\circ}$	0
-1	$C^0_{X^\circ}$	$\mathbb{C}_{X^\circ}$

X =



Upshot: $IC_{X^\circ} = \pi_* \mathbb{C}_{\tilde{X}}[1]$ where

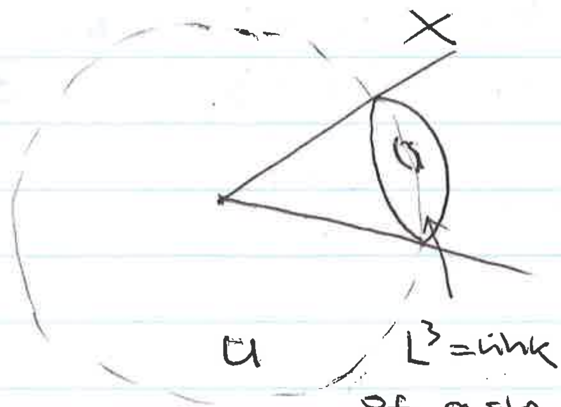
$\pi : \tilde{X} \to X$ is the normalization

---

2) surface with isolated singularities

• $j : X^\circ \hookrightarrow X$ smooth locus.

Construct IC_{X°.



L^3 = link of a sing. pt = 3-mfd

Step 1. Start with $\mathbb{C}_{X^\circ}[2]$

over X°, recover $\mathbb{C}_{X^\circ}[2]$

- Stalk at a sing. pt e of $j_* \mathbb{C}_{X^\circ}[2]$:

$= \Gamma(U, j_* \mathbb{C}_{X^\circ}[2])$

$= C^\cdot(\underbrace{U \cap X^\circ}_{\cong\, \mathbb{R} \times L^3})[2]$

- Calculate coh. of link [2].

$\mathbb{C}_{X^\circ}[2]$:

2	$C^4_{X^\circ}$
1	$C^3_{X^\circ}$
0	$C^2_{X^\circ}$
−1	$C^1_{X^\circ}$
−2	$C^0_{X^\circ} \leftarrow \mathbb{C}_{X^\circ}$

$\rightsquigarrow \Gamma(U, j_* \mathbb{C}_{X^\circ}[2])$:

1	*
0	*
−1	*
−2	*

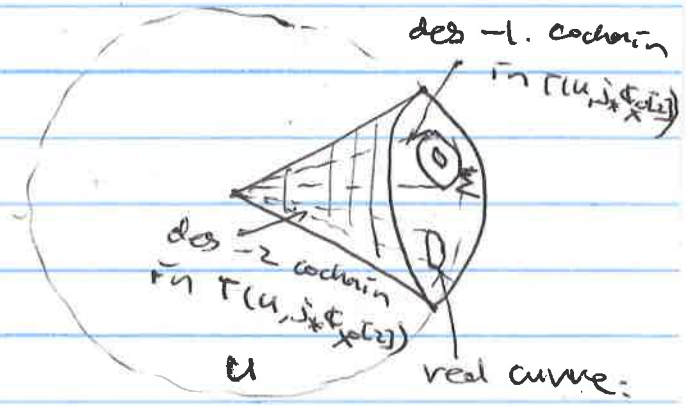
keep. $\to \tau^{\leq -1} \Gamma(U, j_* \mathbb{C}_{X^\circ}[2])$

$\Rightarrow$ Stalk at sing. pt of $\tau^{\leq -1} j_* \mathbb{C}_{X^\circ}[2]$

~~$\tau^{\leq -1}\Gamma(U,)$~~ Calculates H^1, H^0 of L^3.

4.

σ_x = co-stalk of sing. pt of $\tau^{\le -1} j_* \mathbb{C}_{X^\circ}[2]$ calculates H^2, H^3 of L^3.



real curve: deg 0. cochain: not allowed in $\tau^{\le -1} j_*\mathbb{C}_{X^\circ}[2]$

Elementary facts:

i) (topological) $\pi : \tilde{X} \to X$ is a small resolution,

(proper, birational and $\tilde{X}$ is smooth)

stratified:

$\{\tilde{X}_\beta\}_{\beta \in B}$, $\{X_\alpha\}_{\alpha \in A}$

$\pi^{-1}(X_\alpha) = \bigcup_{\beta \in B(\alpha)} \tilde{X}_\beta \longrightarrow X_\alpha$ fibration

2) $\pi: \tilde{X} \to X$ is semismall resolution.

Semismall: For X_α not open,
$$\dim \pi^{-1}(X_\alpha) - \dim X_\alpha \leq \frac{1}{2} \operatorname{codim} X_\alpha .$$

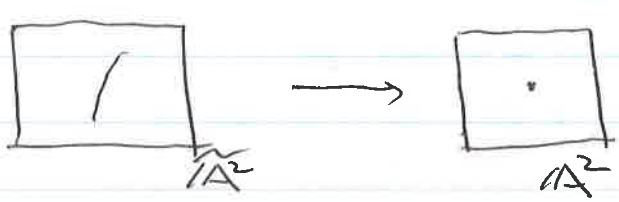
$\Rightarrow$ $\pi_* \mathbb{C}_{\tilde{X}}[\dim \tilde{X}]$ is perverse.

Deep deep theorem (Decomposition Thm)
(Alg. geo., not only topology)

$\pi: \tilde{X} \to X$ proper, ($\tilde{X}$ not nec. smooth)
then:
$$\pi_* IC_{\tilde{X}} = \bigoplus_{(X_\alpha, L_\alpha, d_\alpha)} IC_{X_\alpha}(L_\alpha)[d_\alpha]$$

Examples

1) $\pi: \tilde{\mathbb{A}}^2 \to \mathbb{A}^2$ blow up at 0.
(⇒ semismall resolution)



$$\underset{\substack{\| \\ IC_{\tilde{\mathbb{A}}^2}}}{\pi_* \mathbb{C}_{\tilde{\mathbb{A}}^2}[2]} \simeq \underset{\substack{\| \\ IC_{\mathbb{A}^2}}}{\mathbb{C}_{\mathbb{A}^2}[2]} \oplus \underset{\substack{\| \\ IC_{\{0\}}}}{\mathbb{C}_{\{0\}}} .$$

unshifted IC

complexes. (so reverse)

Ex.: check this iso.

Non-example:

2) $\pi : \mathbb{P}^1 \times \mathbb{A}^1 = \tilde{X} \longrightarrow X = (\mathbb{P}^1\times\mathbb{A}^1)/(\mathbb{P}^1\times\{0\})$

Link $L = \mathbb{C}\mathbb{P}^1 \times S^1$
$= S^2 \times S^1$

Suppose: $\pi_* \mathbb{C}_{\tilde{X}}[2] \cong IC_X \oplus IC_{\{0\}}$

stalk at 0:

	$\pi_* \mathbb{C}_{\tilde{X}}[2]$		IC_X		$IC_{\{0\}}$
2	0		0		0
1	0	$\neq$	0	$\oplus$	0
0	$\mathbb{C}$		0		$\mathbb{C}$
-1	0		$\mathbb{C}$		0
-2	$\mathbb{C}$		$\mathbb{C}$		0

Contradiction!

Cohomology of $\underset{S^2\times S^1}{\overset{\|}{L}}[2]$:

2	0
1	$\mathbb{C}$
0	0
-1	$\mathbb{C}$
-2	$\mathbb{C}$

$\tau^{\leq -1} \rightarrow$

~~Contradiction~~

Correct description. :

$\pi_* \mathbb{C}_{\tilde{X}}[2]$, as a perverse sheaf has a J-H. series with simples:

$$IC_{\{0\}} \quad IC_X \quad IC_{\{0\}}$$

	$\pi_* \mathbb{C}_{\tilde{X}}[2]$	IC_X	$IC_{\{0\}}$	$IC_{\{0\}}$
2	0	0	0	0
1	0	0	0	0
0	$\mathbb{C}$	0	$\mathbb{C}$	$\mathbb{C}$
−1	0	$\mathbb{C}$	0	0
−2	$\mathbb{C}$	$\mathbb{C}$	0	0