We’ll start today by answering a question from last time: why do we have the following equivalence of categories?

\[ \mathcal{D}_c(pt)^T \simeq \text{finitely generated } \Lambda \text{ mod,} \]

where \( T \) is a torus, and \( \Lambda = C_{-\bullet}(T) \simeq \text{Sym}(t[1]) \) with convolution as the multiplication. If \( p : ET \to BT \) is the universal bundle for principal \( T \)-bundles, then an object in \( \mathcal{D}_c(pt)^T \) is a complex \( \mathcal{F}^\bullet \) on \( BT \) such that \( p^*\mathcal{F}^\bullet \) is constant. In other words, \( \mathcal{D}_c(pt)^T \) is equivalent to the derived category of complexes \( \mathcal{F}^\bullet \) on \( BT \) whose cohomology is a local system. **Remark:** \( BT \) is simply connected, so all local systems on it are constant.

Now, consider \( p : ET \to BT \). From this map, we obtain an adjunction:

\[ p_! : \mathcal{D}_c(pt) \leftrightarrow \mathcal{D}_c(pt)^T : p^!, \]

where \( p_! \) is pushforward with compact support and \( p^! \simeq p^*[\text{dim}T], \) since \( T \) is smooth and orientable. We can apply (the derived, infinity-categorical version of) the Barr-Beck theorem to get

\[ \mathcal{D}_c(pt)^T \sim \rightarrow p^!p_! - \text{mod}(\mathcal{D}_c(pt)). \]

Thus, all we have to do is calculate that \( p^!p_! C_{pt} \simeq \Lambda \), and we will have our assertion. Actually, it’s easier to work with the usual \((p^*,p_*)\) adjunction, at the cost of switching from modules over monads to comodules over comonads. The result is that we have

\[ \mathcal{D}_c(pt)^T \sim \rightarrow p^*p_* - \text{comod}(\mathcal{D}_c(pt)). \]

Well, \( p^*p_* C_{pt} \simeq C^\bullet(T) \) as coalgebras, and

\[ C^\bullet(T) - \text{comod}(\mathcal{D}_c(pt)) \simeq C_{-\bullet}(T) - \text{mod}(\mathcal{D}_c(pt)), \]

since \( C_{-\bullet}(T) = C^\bullet(T)^\vee \). As an example, under this equivalence of categories, \( C_{BT} \) goes to \( p^*C_{BT} \simeq C_{pt} \), which is the augmentation module over \( C_{-\bullet}(T) \).

This answers the question from last time. Now, we can move on to studying bases of the Hecke category \( \mathcal{D}_c(G/B)^B \simeq \mathcal{D}_c(G)^{B \times B} \).

1. **Standard Basis:** Let \( w \in \mathcal{W} \) (the Weyl group of \( G \)) and consider the inclusion \( j_w \) of the Schubert cell \( S_w \) into \( G/B \). The standard basis is

\[ \{ J_{w*} = j_{w*}C_{S_w} \mid w \in \mathcal{W} \}. \]

This is a basis in the sense that every object in \( \mathcal{D}_c(G/B) \) is a finite complex built out of \( J_{w*} \)'s.

2. **Costandard Basis:** Define \( J_{w!} \) by replacing \( * \) with \(! \) above.

The issue is that these bases aren’t particularly useful for computation. Theoretically, we have the following theorem:

**Theorem 0.1.** The map \( B\mathcal{W} \to \mathcal{D}_c(G/B)^B \) given by \( s_i \mapsto J_{s_i,*(!)} \) is a homomorphism from the braid group to the Hecke category.

In fact, if \( w_1, w_2 \) are of lengths \( \ell_1, \ell_2 \), respectively, and \( w_1w_2 \) has length \( \ell_1 + \ell_2 \), then

\[ J_{w_1,*(!)} * J_{w_2,*(!)} \simeq J_{w_1w_2,*(!)}. \]
The main problem is that if we remove the condition on the length of $w_1 w_2$, then $J_{w_1, s(t)} \ast J_{w_2, s(t)}$ is very complicated and in particular is not a sum of $J_{w, s}$'s.

As an example, let's take $G = SL(2)$. Then $G \backslash B = \mathbb{P}^1$. Let $j_s : \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ be the inclusion of the affine line into $\mathbb{P}^1$. By definition, $J_{s, s(t)} = j_{s, s(t)} \mathbb{C}_{\mathbb{A}^1}$. We compute:

$$J_{s, s} \ast J_{s, s} = \pi_! \mathbb{C}_X,$$

where

$$X = \{ (\ell_1, \ell_2) \in \mathbb{P}^1 \mid \ell_1 \neq 0, \ell_2 \neq 0 \}$$

($\ell_0$ is a standard line in $\mathbb{A}^2$), and

$$\pi : X \rightarrow \mathbb{P}^1, \quad (\ell_1, \ell_2) \mapsto \ell_2.$$

We find that the stalks of $\pi_! \mathbb{C}_X$ are given by $H^*_c(\mathbb{A}^1)$ over $\infty$ and $H^*_c(\mathbb{G}_m)$ over points in $\mathbb{A}^1$. We might hope that

$$J_{s, s} \ast J_{s, s} = \pi_! \mathbb{C}_X \simeq J_{1!} \otimes H^*_c(\mathbb{A}^1) \oplus J_{s!} \otimes H^*_c(\mathbb{G}_m).$$

But this hope is too optimistic. To see why, we calculate compactly supported global sections of both sides. $\Gamma_c(\mathbb{P}^1, \pi_! \mathbb{C}_X) = H^*_c(X) \simeq \mathbb{C}[-4]$. On the other hand,

$$\Gamma_c(\mathbb{P}^1, J_{1!}) \simeq H^*_c(pt) = \mathbb{C}$$

$$\Gamma_c(\mathbb{P}^1, J_{s!}) \simeq H^*_c(\mathbb{A}^1) = \mathbb{C}[-2].$$

Thus, $J_{1!} \otimes H^*_c(\mathbb{A}^1) \oplus J_{s!} \otimes H^*_c(\mathbb{G}_m)$ is non-zero only in degrees two and three, so we can't have our optimistic hope hold.

Thus, we'd like to find a better basis $\{ I_1, I_s \}$ for the Hecke category. We keep $I_1 = J_{1!} = J_{1*}$, which is our monoidal unit. Now, we take $I_s := \mathbb{C}_{\mathbb{P}^1}[1]$. Note that $I_s$ fits into a triangle

$$J_{1*} \rightarrow I_s \rightarrow J_{ss}[1],$$

which upon rotation becomes

$$I_s \rightarrow J_{ss}[1] \rightarrow J_{1*}[1],$$

so that $J_{ss}$ is built out of the $I$'s. We can also calculate $I_s \ast I_s$. Here, we use

$$\tilde{X} = \{ (\ell_1, \ell_2) \in \mathbb{P}^1 \} = \mathbb{P}^1 \times \mathbb{P}^1$$

and $\pi_2 : \tilde{X} \rightarrow \mathbb{P}^1$, which is simply projection onto the second factor. We compute

$$I_s \ast I_s = \tilde{\pi}_! \mathbb{C}_{\tilde{X}}[2] \simeq \mathbb{C}_{\mathbb{P}^1} \oplus H^*_c(\mathbb{P}^1)[2]$$

$$\simeq \mathbb{C}_{\mathbb{P}^1}[2] \oplus \mathbb{C}_{\mathbb{P}^1}$$

$$\simeq I_s[1] \oplus I_s[-1].$$

This is actually a nice formula for the product of the $I$'s.

Next time, we will introduce the intersection complexes $I_w \in \mathcal{D}_c(G/B)^B$, $w \in \mathcal{W}$ that satisfy the following deep theorem:

$$I_{w_1} \ast I_{w_2} = \bigoplus I_{w_i}[d_i],$$

i.e. taking the monoidal product of these complexes always lands us in sums of the $I_w$'s.