MATH 274 NOTES: 16 MARCH 2017

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We'll start today by answering a question from last time: why do we have the following equivalence of categories?

$$\mathcal{D}_c(pt)^T \simeq \text{finitely generated } \Lambda - mod,$$

where T is a torus, and $\Lambda = C_{-\bullet}(T) \simeq \mathrm{Sym}(\mathfrak{t}[1])$ with convolution as the multiplication. If $p : ET \to BT$ is the universal bundle for principal T-bundles, then an object in $\mathcal{D}_c(pt)^T$ is a complex $\mathcal{F}^\bullet$ on BT such that $p^*(\mathcal{F}^\bullet)$ is constant. In other words, $\mathcal{D}_c(pt)^T$ is equivalent to the derived category of complexes $\mathcal{F}^\bullet$ on BT whose cohomology is a local system. **Remark:** BT is simply connected, so all local systems on it are constant.

Now, consider $p : ET \to BT$. From this map, we obtain an adjunction:

$$p_! : \mathcal{D}_c(pt) \leftrightarrow \mathcal{D}_c(pt)^T : p^!,$$

where $p_!$ is pushforward with compact suppoprt and $p^1 \simeq p^*[dimT]$, since T is smooth and orientable. We can apply (the derived, infinity-categorical version of) the Barr-Beck theorem to get

$$\mathcal{D}_c(pt)^T \xrightarrow{\sim} p^!p_! - mod((\mathcal{D}_c(pt))).$$

Thus, all we have to do is calculate that $p^!p_!\mathbb{C}_{pt} \simeq \Lambda$, and we will have our assertion. Actually, it's easier to work with the usual (p^*, p_*) adjunction, at the cost of switching from modules over monads to comodules over comonads. The result is that we have

$$\mathcal{D}_c(pt)^T \xrightarrow{\sim} p^*p_* - comod(\mathcal{D}_c(pt)).$$

Well, $p^*p_*\mathbb{C}_{pt} \simeq C^\bullet(T)$ as coalgebras, and

$$C^\bullet(T) - comod(\mathcal{D}_c(pt)) \simeq C_{-\bullet(T)} - mod(\mathcal{D}_c(pt)),$$

since $C_{-\bullet}(T) = C^\bullet(T)^\vee$. As an example, under this equivalence of categories, $\mathbb{C}_{BT}$ goes to $p^*\mathbb{C}_{BT} \simeq \mathbb{C}_{pt}$, which is the augmentation module over $C_{-\bullet}(T)$.

This answers the question from last time. Now, we can move on to studying bases of the Hecke category $\mathcal{D}_c(G/B)^B \simeq \mathcal{D}_c(G)^{B\times B}$.

1. *Standard Basis:* Let $w \in \mathcal{W}$ (the Weyl group of G) and consider the inclusion j_w of the Schubert cell S_w into G/B. The standard basis is

   $$\{J_{w*} = j_{w,*}\mathbb{C}_{S_w} \mid w \in \mathcal{W}\}.$$

   This is a basis in the sense that every object in $\mathcal{D}_c(G/B)$ is a finite complex built out of J_{w*}'s.
2. *Costandard Basis:* Define $J_{w!}$ by replacing $*$ with ! above.

The issue is that these bases aren't particularly useful for computation. Theoretically, we have the following theorem:

Theorem 0.1. *The map $B_{\mathcal{W}} \to \mathcal{D}_c(G/B)^B$ given by $s_i \mapsto J_{s_i,*(!)}$ is a homomorphism from the braid group to the Hecke category.*

In fact, if w_1, w_2 are of lengths ℓ_1, ℓ_2, respectively, and w_1w_2 has length $\ell_1 + \ell_2$, then

$$J_{w_1,*(!)} \star J_{w_2,*(!)} \simeq J_{w_1w_2,*(!)}.$$

The main problem is that if we remove the condition on the length of w_1w_2, then $J_{w_1,*(!)} \star J_{w_2,*(!)}$ is very complicated and in particular is not a sum of $J_{w,*}$'s.

As an example, let's take $G = SL(2)$. Then $G\backslash B = \mathbb{P}^1$. Let $j_s : \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ be the inclusion of the affine line into $\mathbb{P}^1$. By definition, $J_{s,*(!)} = j_{s,*(!)}\mathbb{C}_{\mathbb{A}^1}$. We compute:

$$J_{s,!} \star J_{s,!} = \pi_!\mathbb{C}_X,$$

where

$$X = \{\ell_1, \ell_2 \in \mathbb{P}^1 \mid \ell_1 \neq \ell_0, \ell_2 \neq \ell_0\}$$

(ℓ_0 is a standard line in $\mathbb{A}^2$), and

$$\pi : X \to \mathbb{P}^1, \quad (\ell_1, \ell_2) \mapsto \ell_2.$$

We find that the stalks of $\pi_!\mathbb{C}_X$ are given by $H_c^\bullet(\mathbb{A}^1)$ over ∞ and $H_c^\bullet(\mathbb{G}_m)$ over points in $\mathbb{A}^1$. We might hope that

$$J_{s,!} \star J_{s,!} = \pi_!\mathbb{C}_X \simeq J_{1!} \otimes H_c^\bullet(\mathbb{A}^1) \oplus J_{s!} \otimes H_c^\bullet(\mathbb{G}_m).$$

But this hope is too optimistic. To see why, we calculate compactly supported global sections of both sides. $\Gamma_c(\mathbb{P}^1, \pi_!\mathbb{C}_X) = H_c^\bullet(X) \simeq \mathbb{C}[-4]$. On the other hand,

$$\Gamma_c(\mathbb{P}^1, J_{1!}) \simeq H_c^\bullet(pt) = \mathbb{C}$$
$$\Gamma_c(\mathbb{P}^1, J_{s!}) \simeq H_c^\bullet(\mathbb{A}^1) = \mathbb{C}[-2].$$

Thus, $J_{1!} \otimes H_c^\bullet(\mathbb{A}^1) \oplus J_{s!} \otimes H_c^\bullet(\mathbb{G}_m)$ is non-zero only in degrees two and three, so we can't have our optimistic hope hold.

Thus, we'd like to find a better basis $\{I_1, I_s\}$ for the Hecke category. We keep $I_1 = J_{1!} = J_{1*}$, which is our monoidal unit. Now, we take $I_s := \mathbb{C}_{\mathbb{P}^1}[1]$. Note that I_s fits into a triangle

$$J_{1*} \to I_s \to J_{s*}[1],$$

which upon rotation becomes

$$I_s \to J_{s*}[1] \to J_{1*}[1],$$

so that J_{s*} is built out of the I's. We can also calculate $I_s \star I_s$. Here, we use

$$\tilde{X} = \{\ell_1, \ell_2 \in \mathbb{P}^1\} = \mathbb{P}^1 \times \mathbb{P}^1$$

and $\pi_2 : \tilde{X} \to \mathbb{P}^1$, which is simply projection onto the second factor. We compute

$$\begin{aligned} I_s \star I_s &= \tilde{\pi}_!\mathbb{C}_{\tilde{X}}[2] \simeq \mathbb{C}_{\mathbb{P}^1} \oplus H_c^\bullet(\mathbb{P}^1)[2] \\ &\simeq \mathbb{C}_{\mathbb{P}^1}[2] \oplus \mathbb{C}_{\mathbb{P}^1} \\ &\simeq I_s[1] \oplus I_s[-1]. \end{aligned}$$

This is actually a nice formula for the product of the I's.

Next time, we will introduce the intersection complexes $I_w \in \mathcal{D}_c(G/B)^B, w \in \mathcal{W}$ that satisfy the following deep theorem:

$$I_{w_1} \star I_{w_2} = \bigoplus_i I_{w_i}[d_i],$$

i.e. taking the monoidal product of these complexes always lands us in sums of the I_w's.