

Today: Equivariant Constructible Derived Category.

Recall:  $G \curvearrowright X \rightsquigarrow H_G^i(X) = H^i(G \backslash EG \times X)$ .

This can be calculated w.r.t. some cover of  $G \backslash EG \times X$ ; i.e. calculated locally.

$X/G$  has a cover by  $X$ :

$$G \backslash X \leftarrow X \rightrightarrows X \times_{G \backslash X} X \rightrightarrows X \times X \times X \dots$$

$\underbrace{\hspace{10em}}_{\text{simplicial space.}}$

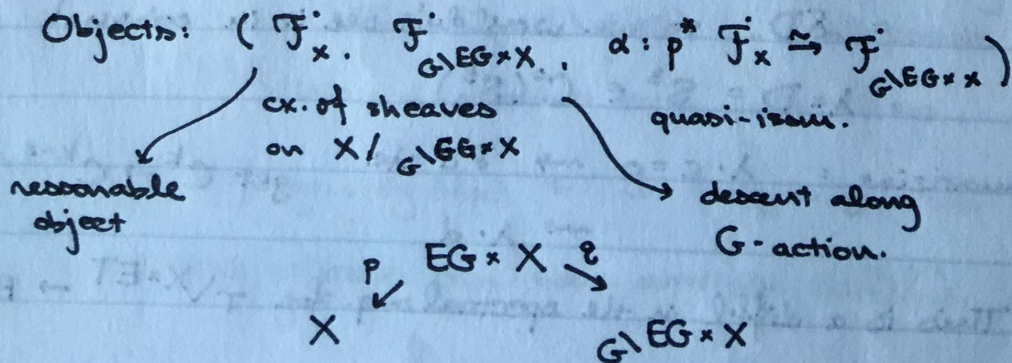
$$C^*(G \backslash X) \xrightarrow{\sim} \text{Tot}(\underbrace{C^*(X \times \dots \times X)}_{\text{cosimplicial complex}})$$

This simplicial space is equiv. to.

$$G \backslash X \leftarrow X \xrightleftharpoons[p]{p} X \times G \rightrightarrows X \times G \times G \dots$$

Conclusion: "cochains on  $G \backslash X$  can be thought of as cochains on  $X$  compatible with pullbacks"

Def:  $G \curvearrowright X$ ,  $D(X)^G$  the equiv. derived cat. consists of



Equivalently, an equivariant complex is a complex  $\mathcal{F}_X$  on  $X$  ( $\mathcal{F}_X$ ) together with compatible identifications for all pullbacks in diagram (given by  $\mathcal{F}_{EG \times X}, \alpha$ ).

$D_c(X)^G \subset D(X)^G$  equivariant constructible der. cat.  
require  $\mathcal{F}_X$  constructible.

$\text{Perv}(X)^G \subset D_c(X)^G$  require  $\mathcal{F}_X$  perverse.

Ex:  $G$  trivial  $\Rightarrow D(X)^G \cong D(X)$ .

Ex:  $G$  contractible  $\Rightarrow D(X)^G \xrightarrow[\text{forget}]{} D(X)$  full subcat.

"equivariant for contractible  $G$  is a property, not a structure"

Ex:  $X = \text{pt}$ .  $D(\text{pt})^G \cong C_*(G)\text{-mod} = C^*(G)\text{-comod}$ .

$\hookrightarrow$  expressing descent for  $\text{pt} \xrightarrow{\pi} BG$ .

[Caution:  $D(\text{pt})^G \neq D(\text{equiv. sheaves})$ .]

Note:  $C^*(G) = \pi^* \pi_* C_{\text{pt}}$  co-monad

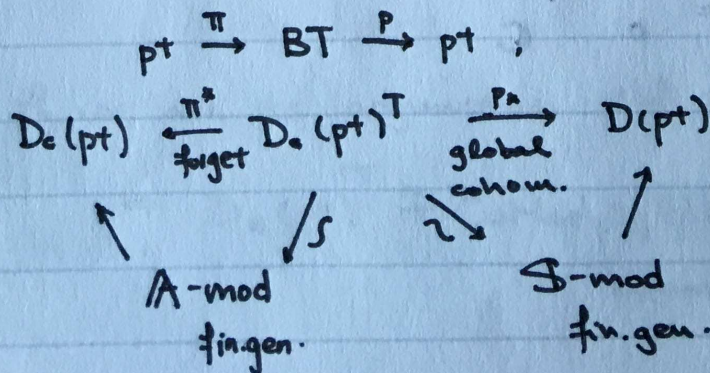
$C_*(G) = \pi^! \pi_! C_{\text{pt}}$  monad.

Ex:  $X = \text{pt}$ .  $G = T$  torus.

$D(\text{pt})^T \cong C^*(T)\text{-comod} \cong \underline{C_*(T)}\text{-mod}$   
 $\cup \quad \quad \quad \wedge$

$D_c(\text{pt})^T \cong \Lambda\text{-dg mod}$ .

Koszul duality:  $D_c(\text{pt})^T \cong \mathbb{S}\text{-dg mod}$ .



Ex:  $B \simeq SL_2/B = \mathbb{P}^1$ .

$$D(\mathbb{P}^1)^B \xrightarrow{\text{forget}} D(\mathbb{P}^1)^T \quad \text{fully faithful}$$

(b.c.  $T \hookrightarrow B$  is homotopic eq).

Recall: 5 indecomp. perv. sheaves constructible w.r.t.

$B$ -orbits.

Now only 4  $B$ -equivariant ones.

- 1).  $\mathbb{C}_{\mathbb{P}^1}[1]$
- 2).  $i_* \mathbb{C}_{\{0\}} = i! \mathbb{C}_{\{0\}} \quad i: 0 \hookrightarrow \mathbb{P}^1$
- 3).  $j_* \mathbb{C}_{A'}[1] \quad j: A' \hookrightarrow \mathbb{P}^1 \xrightarrow{i} 0$
- 4).  $j! \mathbb{C}_{A'}[1]$

Note:  $T$  is not  $B$ -equivariant.  $T$  has J.H. series

$$0 \hookrightarrow \mathbb{C}_0 \hookrightarrow j! \mathbb{C}_{A'}[1] \hookrightarrow T$$

$\mathbb{C}_0 \quad \mathbb{C}_{\mathbb{P}^1}[1] \quad \mathbb{C}_0$  successive quotients.

$\Rightarrow$  SES  $0 \rightarrow j! \mathbb{C}_{A'}[1] \rightarrow T \rightarrow \mathbb{C}_0 \rightarrow 0$ .

Defined by some  $e \in \text{Ext}^1(\mathbb{C}_0, j! \mathbb{C}_{A'}[1])$ .

Show  $e$  is not  $T$ -equiv.

Claim: Ext vanishes in equiv. der. cat.

$$\text{Hom}(\mathbb{C}_0, j! \mathbb{C}_{A'}[1]) \simeq H^*(\mathbb{C}_0, \mathbb{C}_0)[1] = \begin{matrix} 1 & \mathbb{C} \\ 0 & \mathbb{C} \end{matrix}$$

$$\text{Hom}_{\text{Eq. Der.}}(\mathbb{C}_0, \mathbb{C}_0) = H_T^*(\mathbb{C}_0, \mathbb{C}_0)[1]$$

$$= \begin{matrix} 1 & \mathbb{C} \\ 0 & \mathbb{C} \end{matrix}$$