Mar. 14th, 2017.

Today: Equivariant Constructible Derived Category.

Recall: $G \curvearrowright X \rightsquigarrow H^\cdot_G(X) = H^\cdot({}_{G}\backslash EG \times X)$.

This can be calculated w.r.t. some cover of ${}_{G}\backslash EG \times X$; i.e. calculated locally.

X/G has a cover by X:

$$\underbrace{G\backslash X \leftarrow X \leftleftarrows X \times_{G\backslash X} X \Lleftarrow X \times X \times X \cdots}_{\text{simplicial space.}}$$

$\cap$ of opens — triple $\cap$'s.

$$C^\cdot(G\backslash X) \xrightarrow{\sim} \mathrm{Tot}(\underbrace{C^\cdot(X \times \cdots \times X)}_{\text{cosimplicial complex.}}))$$

This simplicial space is equiv. to.

$$G\backslash X \leftarrow X \underset{a}{\overset{p}{\leftleftarrows}} X \times G \Lleftarrow X \times G \times G \cdots$$

Conclusion: "cochains on $G\backslash X$ can be thought of as cochains on X compatible with pullbacks"

· Def. $G \curvearrowright X$, $D(X)^G$ the equivar. derived cat. consists of

Objects: $(\mathcal{F}^\cdot_X, \mathcal{F}^\cdot_{G\backslash EG \times X}, \alpha : p^* \mathcal{F}^\cdot_X \xrightarrow{\sim} \mathcal{F}^\cdot_{G\backslash EG \times X})$

cx. of sheaves on X / ${}_{G}\backslash EG \times X$

quasi-isom.

reasonable object

descent along G-action.

$$X \xleftarrow{p} EG \times X \xrightarrow{q} G\backslash EG \times X$$

Equivalently, an equivariant complex is a complex ~~of~~ on X ($\mathcal{F}_X^\cdot$) together with compatible identifications for all pullbacks in diagram (given by $\mathcal{F}^\cdot_{EG \times_G X}$, α).

$D_c(X)^G \subset D(X)^G$ equivar. constructible der. cat.

require $\mathcal{F}_X^\cdot$ constructible.

$\mathrm{Perv}(X)^G \subset D_c(X)^G$ require $\mathcal{F}_X^\cdot$ perverse.

Ex: G trivial $\Rightarrow$ $D(X)^G \cong D(X)$.

Ex: G contractible $\Rightarrow$ $D(X)^G \underset{\text{forget}}{\hookrightarrow} D(X)$ full subcat.

" equivariant for contractible G is a property, not a structure"

Ex: $X = pt$. $D(pt)^G \simeq C_{-\cdot}(G)$-mod. $\simeq C^\cdot(G)$-comod.

↳ expressing descent for $pt \xrightarrow{\pi} BG$.

[Caution: $D(\cancel{pt}X)^G \neq D(\text{equiv. sheaves})$.]

Note: $C^\cdot(G) = \pi^* \pi_* \mathbb{C}_{pt}$ co-monad

$C_{-\cdot}(G) = \pi^! \pi_! \mathbb{C}_{pt}$. monad.

Ex: $X = pt$. $G = T$ torus.

$$D(pt)^T \cong C^\cdot(T)\text{-comod} \cong \underline{C_{-\cdot}(T)}\text{-mod}$$

$\cup\!|$ $\qquad\qquad\qquad\qquad \Lambda.$

$D_c(pt)^T \simeq \Lambda$-dg mod.

Koszul duality: $D_c(pt)^T \simeq S$ dg-mod.

$$pt \xrightarrow{\pi} BT \xrightarrow{p} pt \ ;$$

$$\begin{tikzcd}
D_c(pt) & D_c(pt)^T \arrow[l, "\pi^*"', "\text{forget}"] \arrow[r, "p_*", "\text{global cohom.}"'] & D(pt) \\
\Lambda\text{-mod fin.gen.} \arrow[u] \arrow[ur, "\wr"] & & S\text{-mod fin.gen.} \arrow[u] \arrow[ul, "\wr"']
\end{tikzcd}$$

Ex: $B \curvearrowright SL_2/B = \mathbb{P}^1$.

$$D(\mathbb{P}^1)^B \xhookrightarrow[\text{forget}]{} D(\mathbb{P}^1)^T \quad \text{fully faithful}$$

(b.c. $T \hookrightarrow B$ is homotopic eq).

Recall: 5 indecomp. perv. sheaves constructible w.r.t. B-orbits.

Now only 4 B-equivariant ones.

1). $\mathbb{C}_{\mathbb{P}^1}[1]$

2). $i_* \mathbb{C}_{\{0\}} = i_! \mathbb{C}_{\{0\}}$. $\quad i : 0 \hookrightarrow \mathbb{P}^1$.

3). $j_* \mathbb{C}_{\mathbb{A}^1}[1]$ $\quad j : \mathbb{A}^1 \hookrightarrow \mathbb{P}^1 \overset{i}{\hookleftarrow} 0$.

4) $j_! \mathbb{C}_{\mathbb{A}^1}[1]$

Note: $\mathbb{T}$ is not B-equivariant. $\mathbb{T}$ has J.H. series

$$0 \hookrightarrow \mathbb{C}_0 \hookrightarrow j_! \mathbb{C}_{\mathbb{A}^1}[1] \hookrightarrow \mathbb{T}$$

$\mathbb{C}_0 \quad \mathbb{C}_{\mathbb{P}^1}[1] \quad \mathbb{C}_0$ successive quotients.

$\Rightarrow$ SES

$$0 \to j_! \mathbb{C}_{\mathbb{A}^1}[1] \to \mathbb{T} \to \mathbb{C}_0 \to 0.$$

Defined by some $e \in \mathrm{Ext}^1(\mathbb{C}_0, j_! \mathbb{C}_{\mathbb{A}^1}[1])$.

Show e is not T-equiv.

Claim: Ext vanishes in equivar. der cat.

$$\mathrm{Hom}(\mathbb{C}_0, j_! \mathbb{C}_{\mathbb{A}^1}[1]) \simeq H^*(\ ,\ )[1] = \begin{array}{cc} 1 & \mathbb{C} \\ 0 & \mathbb{C} \end{array}.$$

$$\mathrm{Hom}_{\text{Eq.Der.}}(\text{''}, \text{''}) = H_T^*(\ ,\ )[1]$$

$$= \begin{array}{ccc} 1 & & 0 \\ 0 & & \mathbb{C} \end{array}$$