Topics in algebra March 8.

$G \circlearrowright X$ (G acts on space X)

Question 1.

Which is smarter (= knows more)

usual cohomology $H^*(X)$ or equivariant cohomology $H^*_G(X)$?

Question 2.

The same about spaces: which is smarter

X or $G \backslash (X \times EG) =: G \backslash\backslash X$?

Lemma

$$\left\{ \begin{array}{l} \text{spaces } X \\ \text{with } G\text{-action} \end{array} \right\} \xrightarrow{\simeq} \left\{ \begin{array}{l} \text{spaces over } BG \\ \quad Y \to BG \\ (\text{fibrations over } BG) \end{array} \right\}$$

$$X \longmapsto (G \backslash\backslash X \to BG)$$

$$Y \times_{BG} EG \longleftarrow\!\shortmid (Y \to BG)$$

Lemma $\Rightarrow$ „neither" is the answer for Question 2.

Back to cohomology

$H_{-\bullet}(G) \curvearrowright H^\bullet(X)$

homology of G — acts on — cohomology of X

"locally constant distributions on G" (algebra with respect to convolution)

module (over this algebra)

$H^\bullet_G(pt) \curvearrowright H^\bullet_G(X)$

algebra with respect to multiplication (i.e. cup product)

module (algebra object)

Examples

$G = T$ (torus), $H_{-\bullet}(T) \simeq \mathrm{Sym}(t[1])$

$\simeq$

0	$\mathbb{C}$
-1	t
-2	$\Lambda^2 t$
$\vdots$	$\vdots$
$-\dim T$	$\Lambda^{\dim T} t \simeq \mathbb{C}$

1) $X = pt$

$H_{-\bullet}(T) \curvearrowright \mathbb{C}$

exterior algebra — augmentation module

$H^\bullet_T(pt) \curvearrowright H^\bullet_T(pt)$

free module

$\simeq \mathrm{Sym}(t^*[-2])$

$\simeq$

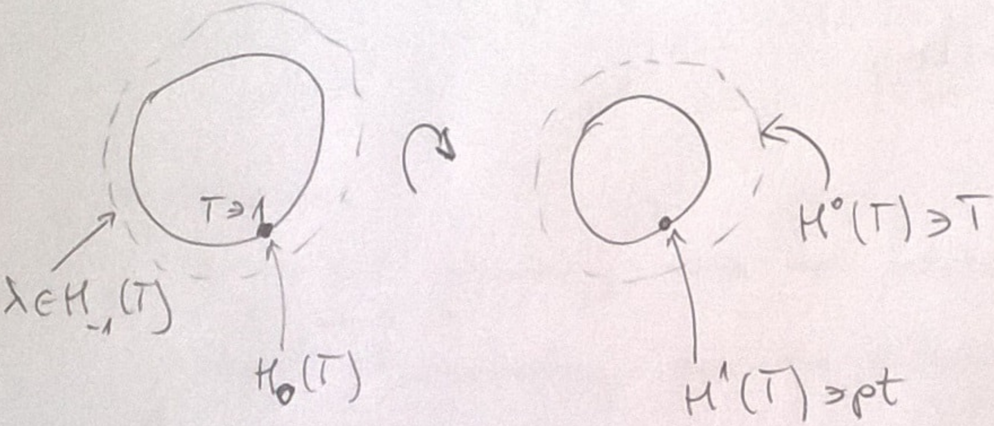
4	$\mathrm{Sym}^2 t^*$
2	t^*
0	$\mathbb{C}$

2) $X = T$

$$H_{-\bullet}(T) \curvearrowright H^\bullet(T) \qquad\qquad H^\bullet_T(pt) \curvearrowright H^\bullet_T(T) \cong \mathbb{C}$$

$$\mathrm{Sym}(\mathfrak{t}^*[-1]) \xrightarrow{\text{free module}} H^\bullet(T)$$

augmentation module



$$\lambda * pt = T$$

free module given by swiping the functions

3) $X = S^3 \circlearrowleft T = S^1$ (Hopf action), $\quad X/_T \simeq S^2$

$X' = S^2 \times S^1 \circlearrowleft T = S^1$ (diagonal action), $\quad X'/_T \simeq S^2$

X:

$$H_{-\bullet}(S^1) \curvearrowright H^\bullet(S^3) \qquad\qquad H^\bullet_{S^1}(pt) \curvearrowright H^\bullet_{S^1}(S^3) \simeq H^\bullet(S^2)$$

$$\begin{array}{cc} 0 & \mathbb{C} \\ -1 & \mathbb{C} \end{array} \qquad \begin{array}{cc} 3 & \mathbb{C} \\ 2 & 0 \\ 1 & 0 \\ 0 & \mathbb{C} \end{array} \qquad \begin{array}{cc} \vdots & \vdots \\ 4 & \mathbb{C} \\ 2 & \mathbb{C} \\ 0 & \mathbb{C} \end{array} \qquad \begin{array}{cc} 2 & \mathbb{C} \\ 0 & \mathbb{C} \end{array}$$

"$=1$" "$=\lambda$"

λ acts trivially

$$k[u] \curvearrowright k[u]/(u^2)$$

X':

$$H_{-\bullet}(S^1) \curvearrowright H^*(S^2 \times S^1) \qquad H^\bullet_{S^1}(pt) \curvearrowright H^\bullet_{S^1}(S^2 \times S^1) \cong H^\bullet(S^2)$$

$$\begin{array}{ll} 0 & \mathbb{C}\,1 \\ -1 & \mathbb{C}\,\lambda \end{array} \qquad \begin{array}{ll} 3 & \mathbb{C} \\ 2 & \mathbb{C} \\ 1 & \mathbb{C} \\ 0 & \mathbb{C} \end{array} \begin{array}{l} \big\}\lambda \\ \\ \big\}\lambda \end{array} \qquad \begin{array}{ll} \vdots & \vdots \\ 4 & \mathbb{C} \\ 2 & \mathbb{C} \\ 0 & \mathbb{C} \end{array} \qquad \begin{array}{ll} 2 & \mathbb{C} \\ 0 & \mathbb{C} \end{array}$$

$$\lambda^2 = 0 \qquad k[u] \curvearrowright k[u]/(u^2)$$

Conclusion:

equivariant cohomology does not catch differences between X and X',
usual cohomology sees the difference between X and X'.

~~needs~~ correction is needed (to help equivariant cohomology):
do not pass to cohomology and homology,
but work with cochains and chains complexes.

Koszul duality: $\Lambda = \mathrm{Sym}(V[1])$, $S = \mathrm{Sym}(V^*[-2])$
V: finite dimensional vector space

Thm 1.

$$\{\Lambda\text{-finitely generated dg-modules}\} \simeq \{S\text{-finitely generated dg-modules}\}$$

Thm 2.

$T \curvearrowright X$ (reasonable)

$$C^\bullet(X) \longleftrightarrow C^\bullet_T(X) \quad \text{[under Thm 1.]}$$

$M \longmapsto \mathrm{Hom}_\Lambda(\mathbb{C}, M)$, Λ-invariant, augmentation module

$\mathrm{Hom}_{C_{-\bullet}(T)}(C^\bullet(ET), C^\bullet(X)) = C^\bullet_T(X)$

$$\mathbb{C} \otimes_S N \longleftarrow N$$

augmentation module

$$C^\bullet(X) = C^\bullet(ET) \otimes_{C_T^\bullet(pt)} C_T^\bullet(X)$$

Key point

$$C_{-\bullet}(T) \curvearrowright C^\bullet(ET) \curvearrowleft C_T^\bullet(pt)$$

bimodule

each is endo of module over other

Abstractly

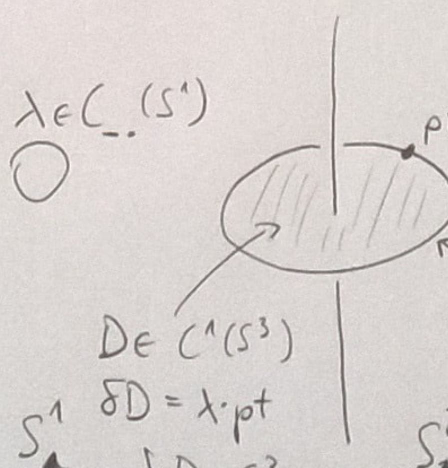
$$\Lambda \curvearrowright \mathbb{C} \curvearrowleft S$$

trivial as left or right module, Koszul bimodule as bimodule

Back to X and X'

$$C_{-\bullet}(T) \curvearrowright C^\bullet(X)$$

is not formal (not equivalent to passing to cohomology)



$\lambda \in C_{-\bullet}(S^1)$

$pt \in C^3(S^3)$

$\lambda \cdot pt \in C^2(S^3)$

$D \in C^1(S^3)$

$\delta D = \lambda \cdot pt$

$\lambda \cdot D = S^3 \in C^0(S^3)$

S^1 S^3

Nontrivial secondary operation:

of degree -3

$c \in C^k(X)$ $\quad \lambda \cdot c = 0 \rightsquigarrow \delta d = \lambda c$

$\rightsquigarrow \lambda d \in C^{k-3}(X)$

This is a differential is spectral sequence for $\left(T^{\wedge (X \times ET)} \longrightarrow BT \right)$