3/7/2017

Equivariant localization.

• Equivariant localization (for tori)

$T \underset{*}{\simeq} (\mathbb{C}^*)^n \curvearrowright X$ reasonable.

(Equivariant coh. only depends on the homotopy type of the group)

Thm. $H_T^\cdot(X) \longrightarrow H_T^\cdot(X^T) \simeq H^\cdot(X^T) \otimes H_T^\cdot(pt)$ is an isomorphism, modulo $H_T^\cdot(pt)$-torsion.

$\left(H_T^\cdot(pt) = \mathrm{Sym.}(\mathfrak{t}^*[-2]) \simeq \mathbb{C}[u_1, .., u_n],\ |u_i| = 2 \right)$.

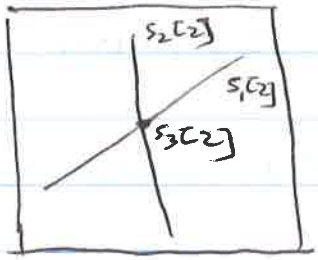
• Equivalently, $H_T^*(X, X^T)$ is $H_T^\cdot(pt)$-torsion.

($H_T^\cdot(pt)$-torsion means supp. away from generic pt. of $\mathrm{spec}\, H_T^\cdot(pt) = \mathfrak{t}[2]$)

• Idea of pf.

• In general, $H_T^\cdot(X)$, $H_T^\cdot(X, Y)$ are supported in $\bigcup_{\substack{\text{stabilizer } S \\ \text{of orbits} \\ \text{in } X \text{ or } X \setminus Y}} \mathrm{spec}.\, H_S^\cdot(pt) \hookrightarrow \mathrm{spec}\, H_T^\cdot(pt)$

picture.



$t[z]$

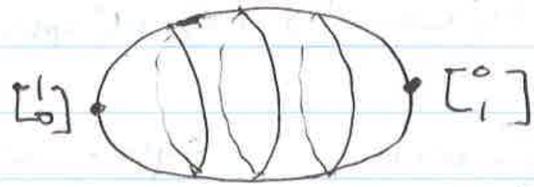
• Reduces to special case.

Stab. $S \hookrightarrow T \curvearrowright Q$ orbit, $H_T^{\cdot}(Q) \overset{Ex.}{\simeq} H_S^{\cdot}(pt)$.

$\square$

Example: 1). $SL(2) \underset{\substack{\text{max.}\\\text{torus}}}{\supset} T \curvearrowright SL(2)/B = \mathbb{P}^1$.

$$\begin{bmatrix} z & \\ & z^{-1} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} zx \\ z^{-1}y \end{bmatrix}.$$



open. orbit T/μ_2.

Equivariant localization $\Rightarrow$ $H_T^{\cdot}(\mathbb{P}^1) \to \overset{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{H_T^{\cdot}(pt)} \oplus \overset{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{H_T^{\cdot}(pt)}$

$\qquad\qquad\qquad\qquad \| \qquad\qquad \|$

becomes isom. over $\mathbb{C}[u, u^{-1}]$. $\qquad \mathbb{C}[u] \oplus \mathbb{C}[v]$

$|u| = |v| = 2$

$$\begin{array}{ll} \vdots & \\ 4 & \mathbb{C}u^2 \oplus \mathbb{C}v^2 \\ 2 & \mathbb{C}u \oplus \mathbb{C}v \\ 0 & \mathbb{C}1 \oplus \mathbb{C}1 \end{array}$$

$pt = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \hookrightarrow \mathbb{P}^1 \hookrightarrow (\mathbb{P}^1, pt)$.

$\rightsquigarrow$

$$\xleftarrow{[1]} \underset{\mathbb{C}[u]}{\overset{H_T^*(pt)}{\parallel}} \leftarrow H_T^\cdot(\mathbb{P}^1) \leftarrow v\,\mathbb{C}[v] \leftarrow$$

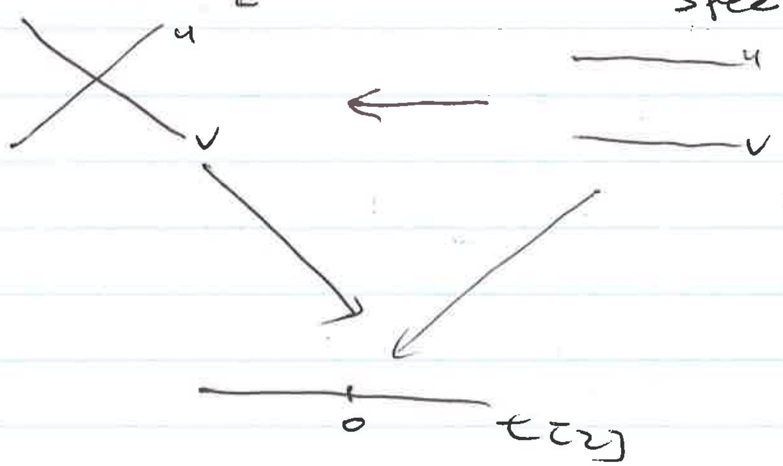
$(|v|=2)$

$\vdots$		$\vdots$	
$4:$	$\mathbb{C}u^2$	$\mathbb{C}u^2 \oplus \mathbb{C}v^2$	$4: \mathbb{C}v^2$
$2:$	$\mathbb{C}u$	$\mathbb{C}u \oplus \mathbb{C}v$	$2: \mathbb{C}v$
$0:$	$\mathbb{C}1$	$\mathbb{C}\cdot 1$	

view as q.c. sheaves over $t[z]$.

spec$(\mathbb{C}[u] \oplus \mathbb{C}[v])$

str. sheaf of $\{uv=0\}$



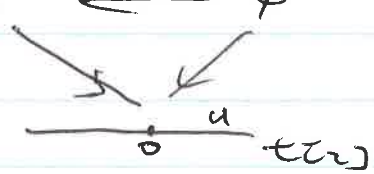
triv. 2. free. str. sheaf of unrami. 2→1 cover

- Equivariant localization gives isom. over $t[z] \setminus \{0\}$.

2) $T \curvearrowright \underset{X}{\overset{T}{\parallel}}$ freely $\Rightarrow X^T = \phi$.

$\rightsquigarrow \underset{\mathbb{C}}{\overset{H_T^\cdot(T)}{\parallel}} \longrightarrow H_T^\cdot(\phi) = 0$

$\rightsquigarrow$ spec $H_T^\cdot(T) = \bullet \longleftarrow \phi = \mathrm{spec}\, H_T^\cdot(\phi)$.



4.

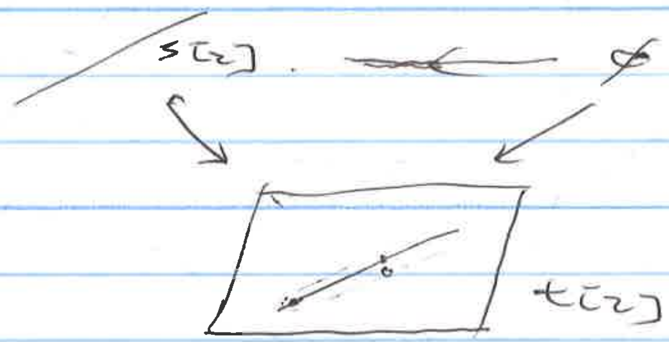
· Equivariant localization gives isom. over $\mathfrak{t}[z]\setminus\{0\}$.

2'). $T \curvearrowright Q$ transitively $\Rightarrow X^T = \phi$.

S = stabilizer

$H_S^\cdot(pt) = H_T^\cdot(Q) \longrightarrow H_T^\cdot(\phi) = (0)$

2).



· Equivariant localization gives isom. over $\mathfrak{t}[z]\setminus \mathfrak{s}[z]$.

3). $T \curvearrowright SL(2) \curvearrowleft T$.

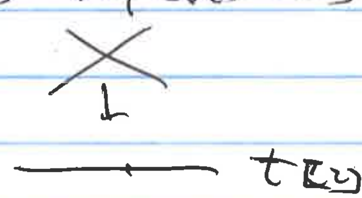
maximal torus

· $H_{T\times T}^\cdot(SL(2)) \simeq H_{T\times\{1\}}(SL(2)/T)$

h.e. $\mathbb{P}^1$

In 1), we said $H_T^\cdot(\mathbb{P}^1)$ as $H_T^\cdot(pt)$-algebra

(spec:



)

Here we want: $H^\bullet_{T\times T}(SL(2))$ as a $H^\bullet_{T\times T}(pt)$-algebra.

$$\begin{array}{ccl}
t[z] \ (\text{right } T) & \longleftarrow & \operatorname{spec} H^\bullet_{T\times T}(SL(2)) = \bigcup_{w\in W} \Gamma_w \\
 & & \big\downarrow \\
 & & t[z] \ (\text{left } T)
\end{array}$$

(Γ_w: graph for $W \curvearrowright T \rightsquigarrow W \curvearrowright H^\bullet_T(pt)$)

"Morse theory"

<u>Refined equivariant localization</u> for <u>equiv formal</u> actions.

<u>Recall</u>. $T \curvearrowright X$ alg. variety,

<u>Assume</u>: 0) finitely many fixed pts

1) finitely many 1-dim orbits. ($\mathbb{G}_m \simeq \mathbb{P}^1 \setminus \{0,\infty\}$)

$\rightsquigarrow$ Spectral sequence:

$$H^\bullet(BT, H^\bullet(X)) \Longrightarrow H^\bullet_T(X)$$

<u>Equiv. formal</u> means: S.S. degenerates ~~at~~ at E_2.

For example, if $H^*(X)$ is concentrated in <u>even degrees</u>.

<u>Then</u>: $0 \to H_T^*(X) \xrightarrow{r^*} \Big[\bigoplus_{q \in \text{fixed pts}} H_T^*(q) \xrightarrow{\pi_0^* - \pi_\infty^*} \bigoplus_{L \in 1\text{-dim orbits}} H_T^*(L) \Big]$

is exact.

<u>where</u>: $\underset{\mathbb{P}^1 \setminus \{0,\infty\}}{L \underset{k}{\cong} \mathbb{G}_m} \begin{array}{l} \xrightarrow{\pi_0} 0 \\ \xrightarrow{\pi_\infty} \infty \end{array} \rightsquigarrow \pi_0^*, \pi_\infty^*$.

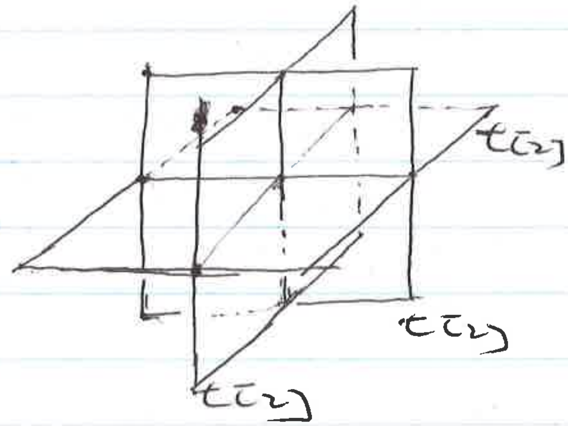
<u>Example</u>. 1) $X = \mathbb{P}^2 \circlearrowleft T \underset{\substack{\text{max} \\ \text{torus}}}{\subset} SL(3)$.

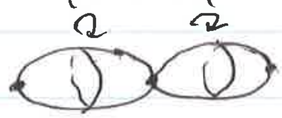
moment map picture for $T \curvearrowright \mathbb{P}^2$

Refined equiv. localization $\rightsquigarrow$

$$\operatorname{Spec} H_T^*(X) \underset{\text{dominant}}{\leftarrow} \bigsqcup_{\text{fixed pts}} \overset{\text{affine}}{\mathfrak{t}[z]} \underset{\text{gluing}}{\leftleftarrows} \bigsqcup_{\text{1-dim orbits}} \overset{\text{codim 1- affine spaces}}{\mathfrak{s}_i[z]}$$

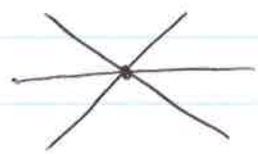
$\Rightarrow \operatorname{Spec} H^*_T(\mathbb{P}^2) =$

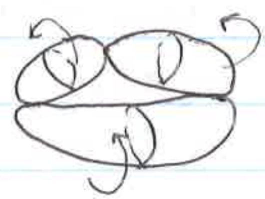


2) $X =$ 

$T = \mathbb{G}_m$

$\mathbb{P}^1 \vee_{pt} \mathbb{P}^1$.

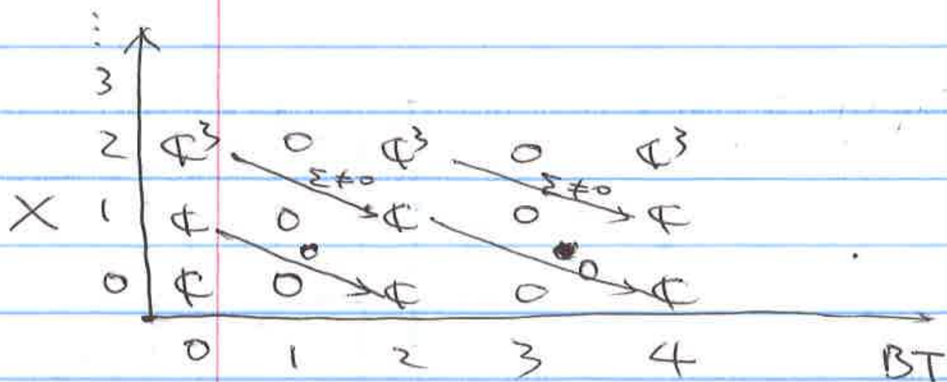
$\Rightarrow \operatorname{Spec} H^*_T(X) =$ 

<u>Non-example</u>. $X \circlearrowleft T = \mathbb{G}_m$ $=$ 

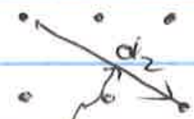
<u>Note</u>. $H^1(X) = \mathbb{C} \neq 0$.

($\rightsquigarrow$ has a chance to be <u>not</u> equiv. formal. Indeed, it's not.)

$$H^*(X) = \begin{cases} \mathbb{C}, & * = 0, 1 \\ \mathbb{C}^3, & * = 2 \end{cases}$$

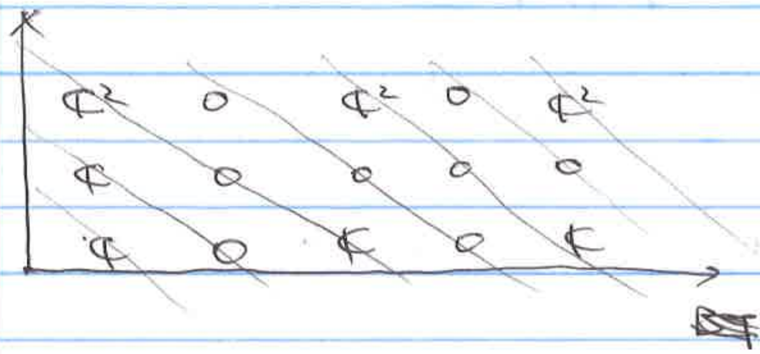


E_2 :



$C_\cdot(T) \circlearrowright C^*(X)$

$\rightsquigarrow$



E_3 -

$\vdots$

5 : 0

4 : $\mathbb{C}^3$

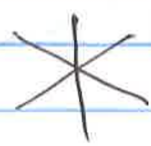
3 : 0

2 : $\mathbb{C}^3$

1 : $\mathbb{C}$

0 : $\mathbb{C}$.

$\Rightarrow$ Not equiva. formal !

Spec $\neq$  (Expected by Thm).