Langlands Duality Lecture 12

March 2, 2017

Last time we constructed the Hecke algebra H_q as the $\mathbb{C}$-valued functions on $B\backslash G/B(\mathbb{F}_q)$ with the algebra structure given by convolution, and we gave a topological interpretation for the case $q = 1$. Our next goal will be to study the Hecke category, which will be the derived category of sheaves of $\mathbb{C}$-vector spaces on $B\backslash G/B$. Thinking of the double coset space as $B\backslash(G/B)$, this is the same as studying B-equivariant sheaves on the flag variety. Therefore we will need to talk about the equivariant derived category of sheaves, and as a first step in that direction we will review equivariant cohomology.

Recall that if X is a space and $\mathbb{C}_X$ is the constant sheaf on X, then we have an algebra isomorphism

$$\mathrm{End}_{D(X)}(\mathbb{C}_X) = H^*(X, \mathbb{C})$$

were $D(X)$ denotes the derived category of X. Indeed, to compute the endomorphisms we may take the resolution $\mathbb{C}_X \to C_X^\bullet$, where $C_X^\bullet$ denotes cochains on X. Therefore $\mathrm{End}_{D(X)}(\mathbb{C}_X) = \mathrm{Hom}(\mathbb{C}_X, C_X^\bullet) = \Gamma(X, C_X^\bullet)$, which computes the cohomology of X.

Now, since $\mathbb{C}_X$ is a unit of the monoidal structure on $D(X)$, we get that $\mathrm{End}_{D(X)}(\mathbb{C}_X) = H^*(X, \mathbb{C})$ acts as endomorphisms of the identity functor of $D(X)$. In the equivariant case, we expect the equivariant cohomology to give us endomorphisms of the identity functor of the equivariant derived category.

Primer on Equivariant Cohomology

Let X be a space, and G be a topological group acting on X. As a first approximation, one may difine the equivariant cohomology $H_G^*(X)$ as the cohomology of X/G. However this is not well behaved when the action is not free, and by taking the naive quotient we are throwing away the information contained in the stabilizers. A better notion comes from first replacing X by a homotopy equivalent space with a free action. We therefore define the equivariant cohomology $H_G^\bullet(X) = H^\bullet(G\backslash(EG \times X))$, where EG is a contractible space with a free G action. This may be seen to be independent of the choice of EG.

We may also give an explicit construction of EG. In general, given Y a (reasonable) space, we may construct a simplicial space whose space of n-simplices is Y^n. Then its geometric realization EY may be seen to be contractible.

More explicitly, EY is built out of Y by first adding a path joining each pair of points, then filling the new triangles determined by those paths, etc. In the case where $Y = G$, the group G acts on EG, and this action may be seen to be free.

Examples:

1. If $Y = *$ then $EY = *$.
2. If $Y = S^0$ then $EY = S^\infty \subset \mathbb{R}^\infty$.
3. If $Y = S^1$ then $EY = S^\infty = (\mathbb{C}^\infty)^\times/\mathbb{R}^\times$.

Now, define $BG = EG/G$. This is the classifying space for principal G-bundles. By definition, the equivariant cohomology of the point $H^*_G(*)$ is equal to $H^*(BG)$. This is the base ring for equivariant cohomology: if $G \curvearrowright X$ then we have a fibration

$$G\backslash(X \times EG) \to G\backslash EG = BG$$

with fiber X, which induces a map of algebras $H^*_G(*) \to H^*_G(X)$. Moreover, we have the Leray spectral sequence $E_2 = H^*(BG, H^*(X)) \Rightarrow H^*_G(X)$. We will say that the action of G on X is equivariantly formal if the spectral sequence degenerates at E_2.

Examples

Observe that if we have a morphism $G \to H$ which is a homotopy equivalence and $H \curvearrowright X$, then G also acts on X and the induced map $H^*_H(X) \to H^*_G(X)$ is an isomorphism. With that in mind, we do some examples

1. Let $T = S^1 \simeq \mathbb{C}^\times$. Then $H^*_T(*) = H^*(\mathbb{CP}^\infty) = k[u]$ where $\deg u = 2$. We think of that ring as being the ring of functions on the affine line.
2. Let $T \subset G = SU(n) \simeq SL(n,\mathbb{C})$ be the maximal torus. Then $H^*_T(*) = k[u_1, \ldots, u_{n-1}]$. Moreover, the map $H^*_G(*) \to H^*_T(*)$ may be seen to be an injection, and in fact it identifies $H^*_G(*)$ with the ring of invariants $k[u_1, \ldots, u_{n-1}]^W$.

Equivariant Localization

Let $T = (S^1)^n \simeq (\mathbb{C}^\times)^n$. Let $T \curvearrowright X$ be an action, and X^T be the fixed points.

Theorem. *The map $H^*_T(X) \to H^*_T(X^T) = H^*(X^T) \otimes H^*_T(*)$ becomes an isomorphism after tensoring with $k(u_1, \ldots, u_n)$ (where the tensor product is over $H^*_T(*) = k[u_1, \ldots, u_n]$).*