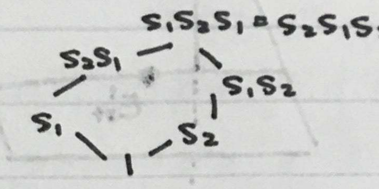
Feb. 28^{th}. 2017.

Warmup: example of a singular Schubert variety
(closure of Schubert cell).

Recall: $X = G/B$ flag variety.

$$= \coprod_{w \in W} X_w.$$

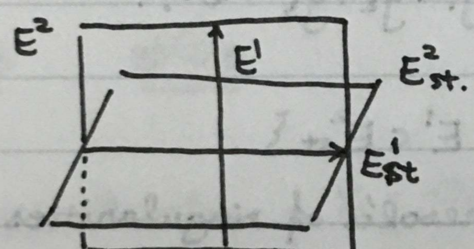
Ex: $G = SL_3$. Cayley graph
(GL_3).



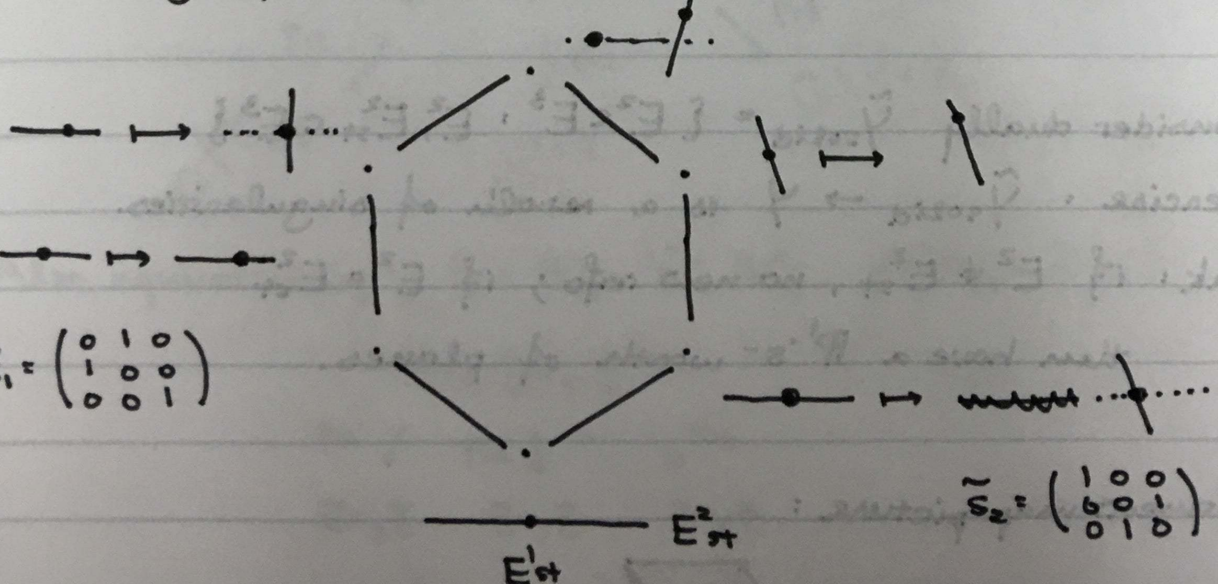
Fix the standard flag. $E^{\cdot}_{st} = 0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \mathbb{C}^3$.

$X_w = \{ E^{\cdot} \in X : \text{rel. pos}(E^{\cdot}, E^{\cdot}_{st}) = w \}$.

Take $w = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, then



Hexagon picture. (projective picture)



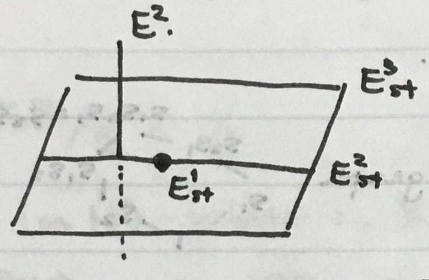
$\tilde{s}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\tilde{s}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Ex: $G = GL_4$ "first" singular Schubert variety.

Consider the partial flag variety $Gr(2,4)$.

(Projective picture).



Consider $Y = \{ E^2 \in Gr(2,4) : E^2 \cap E^2_{st} \text{ has dim} \geq 1 \}$.

Exercise: $\dim Y = 3$, and is smooth away from
$E^2 = E^2_{st}$.
(locally: $y_0^2 + y_1^2 + y_2^2 + y_3^2 = 0$).

Consider $\tilde{Y} = \{ E^1 \subset E^2 : E^1 \subset E^2_{st} \}$.

Exercise: $\tilde{Y} \to Y$ is a resol'n of singularities.

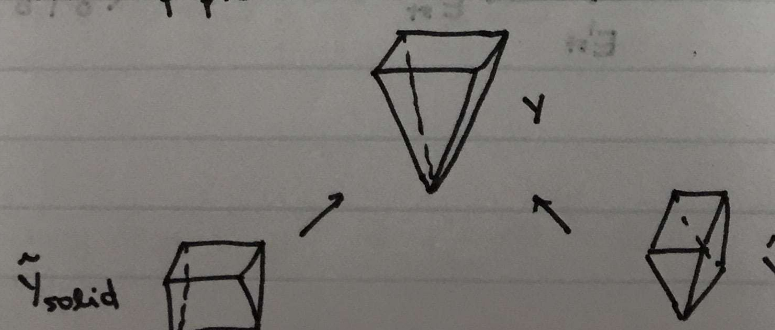
Rmk: if $E^2 \neq E^2_{st}$, no new info; if $E^2 = E^2_{st}$,
then need to pick a line in E^2_{st}.

Consider dually $\tilde{Y}_{solid} = \{ E^2 \subset E^3 : E^2, E^2_{st} \subset E^3 \}$

Exercise: $\tilde{Y}_{solid} \to Y$ is a resol'n of singularities.

Rmk: if $E^2 \neq E^2_{st}$, no new info; if $E^2 = E^2_{st}$
then have a $\mathbb{P}^1$'s-worth of planes.

Moment map picture:



- Back to Convolution. (on $B\backslash G/B$).

G grp.

$$\begin{tikzcd} & G \times G \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[r, "\mu"] & \underline{G} \\ \underline{G} & \underline{G} & \end{tikzcd}$$

can "pull back" via π_1 & π_2 and pushforward via μ, to get " "
"fn's on G" $\times$ "fn's on G" $\to$ "fn's on G".

$B \subset G$ subgrp. picture

$$\begin{tikzcd} & B\backslash G \times_B G/B \arrow[dl] \arrow[dr, "\mu"] & \\ B\backslash G/B \times B\backslash G/B \arrow[d] \arrow[dr] & & \underline{B\backslash G/B} \\ \underline{B\backslash G/B} & \underline{B\backslash G/B} & \end{tikzcd}$$

Exercise. Equivalently. $X = G/B$. $B\backslash G/B = G\backslash X \times X$
↳ diag. action

$$\begin{tikzcd} & G\backslash X \times X \times X \arrow[dl, "p_{12}"'] \arrow[d, "p_{23}"] \arrow[dr, "p_{13}"] & \\ G\backslash X \times X & G\backslash X \times X & G\backslash X \times X. \end{tikzcd}$$

Also equivalently: $Z = BB$, $Y = BG$, $B\backslash G/B = Z \times_Y Z$.

$$\begin{tikzcd} & Z \times_Y Z \times_Y Z \arrow[dl, "p_{12}"'] \arrow[d, "p_{23}"] \arrow[dr, "p_{13}"] & \\ Z \times_Y Z & Z \times_Y Z & Z \times_Y Z \end{tikzcd}$$

if Z is a fin set, $Y = pt$, this gives matrix mult.

Now "linearize": need pullback:

multiply / intersect

pushforward / integrate.

(remark: associativity requires base change).

Consider working over $\mathbb{F}_q$ instead of $\mathbb{C}$.

$G(\mathbb{F}_q) \supset B(\mathbb{F}_q)$. $X(\mathbb{F}_q) = G/B(\mathbb{F}_q)$. fin. sets!

· Def: $\mathcal{H}_q = \mathbb{C}[B\backslash G/B(\mathbb{F}_q)]$ Hecke alg. with convolution.

$\dim \mathcal{H}_q = |W|$ ($= n!$ for $G = GL_n$).

Basis: $T_w = \chi_{X_w} = \begin{cases} 1 & \text{in } X_w \\ 0 & \text{else.} \end{cases}$

Exercise: T_1 = unit of $\mathcal{H}_q$, T_{s_i}'s generate.

Exercise: Serre relations: $T_{s_i} T_{s_{i+1}} T_{s_i} = T_{s_{i+1}} T_{s_i} T_{s_{i+1}}$.

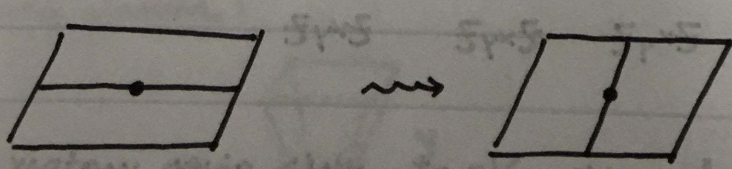
T_{s_i} & T_{s_j} commute if not neighbors.

Calculate $T_{s_i}^2 = ?$

X_{s_i} = pairs of flags

$= \{ E_1^\cdot, E_2^\cdot : E_1^j = E_2^j, E_1^i \neq E_2^i \; j \neq i \}$

= "move i^{th} subspace".



$$T_{s_i}^2(X_w) = \begin{cases} 0 & \text{if } w \neq 1, s_i. \\ q = \# \mathbb{A}^1_{\mathbb{F}_q} & \text{if } w = 1 \\ q-1 & \text{if } w = s_i \end{cases}$$

$$= q + (q-1)T_{s_i}$$

$\Rightarrow \mathcal{H}_q = \mathbb{C}\langle T_{s_i}\text{'s}\rangle / (\text{Serre relation}, T_{s_i}^2 = q + (q-1)T_{s_i})$.

Specialize to $q = 1$. $\rightsquigarrow$ $\mathcal{H}_1$ = grp alg. of Weyl grp.

$$T_{s_i}^2(X_w) = \begin{cases} 0 & \text{if } w \neq 1, s_i \\ 1 = \chi(\mathbb{A}^1_{\mathbb{C}}) & \text{if } w = 1 \quad \text{(Euler char).} \\ 0 = \chi(\mathbb{G}_m) & \text{if } w = s_i \end{cases}$$