02.23.2017.

Langlands duality

Lec. 10.

<u>Old focus</u> geom of reps

spec hw reps. of $\mathfrak{sl}(2)$.

<u>New focus</u>. Hecke categories

Spec finite Hecke cat. of $SL(n)$

<u>Relation between topics</u> (informal rmk)

G gp. rep.:

$\cup$

H subgp. $\quad \mathrm{Ind}_H^G(\mathrm{Triv}) =$ "linearization" of G/H

$G \circlearrowright$

Old: $H = B$

Mata Thm

End_G("linearization" of G/H) = "Linearization" of ${}_H\backslash G/_H$

↑ Induced rep.

Convolution product ↑ Hecke algebra

Old: $G/_B$

New: $B\backslash G/_B$

ind → decomp. → find irr.

Basic terminology

Borel $\quad N = \{$ strict upper Δ-mat. unipotent $\}$

$\mathfrak{g} = \mathfrak{sl}(n)$. $\quad G = SL(n) \supset B = \{$upper Δ matrices$\}$, $\quad B \supset N$

conn. red. (for G)

$B \supseteq T$ max. torus $\{$diag. matrix$\}$

$T \xrightarrow{\cong} H = B/N$ univ. Cartan

$N_G(T) =$ norm of T $\quad \begin{pmatrix} 0 & 0 & * & 0 \\ * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$

Facts.
1) All Borels are conj.
2) Stab of given Borel is that Borel

3) All maximal tori are conj.

It is unamb. (indep. of B) $\quad \nexists$ amb. up to W.

$W = N_G(T)/T = \Sigma_n$ = permutation group

Weyl gp. (Caution: not a subgp. of G)

Weights/chars. $\Lambda_T^\vee = X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$

Coweights/cochars. $\Lambda_T = X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$

$R \subset \Lambda_T^\vee$ roots

$G \circlearrowright \mathfrak{g}$ adj. rep.

$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R^{pos}} \mathfrak{b}_\alpha$

<u>Restrict to $T \subset G$</u>

lines

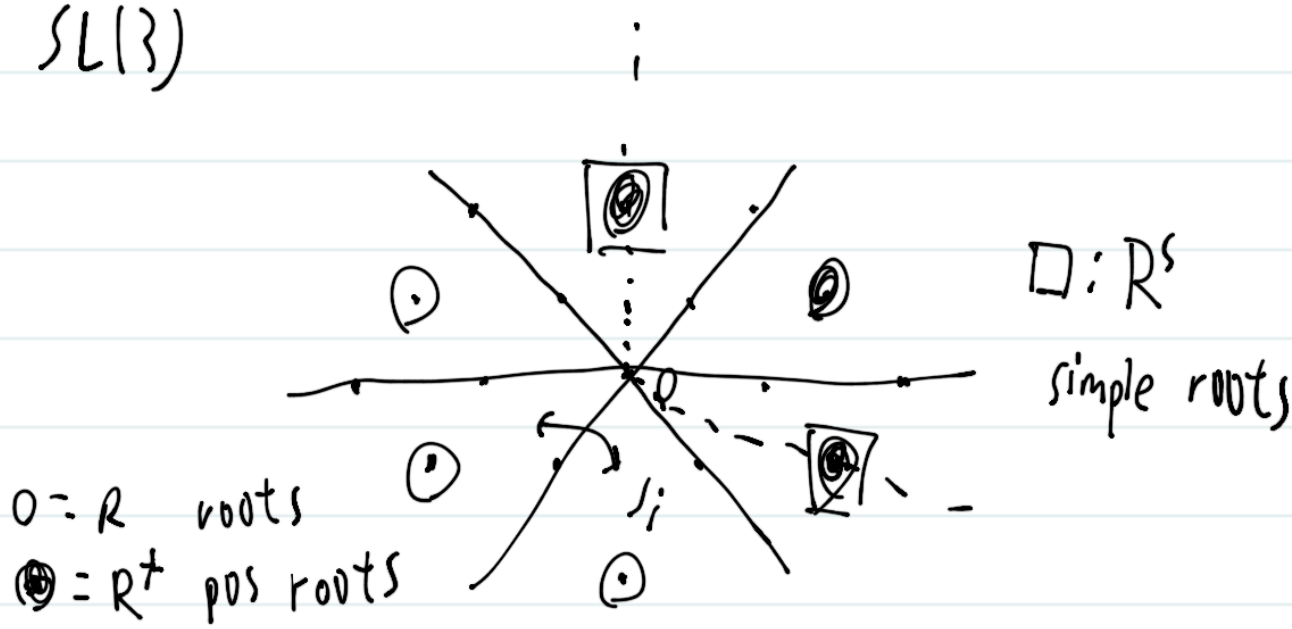
$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$

$\mathfrak{g}_\alpha = \mathbb{C} \cdot E_{ij}$ $\quad \alpha_{ij} = z_i z_j^{-1}$.

$i \neq j$

<u>Ex</u>. $SL(3)$

$\cdot = \Lambda_T^\vee$

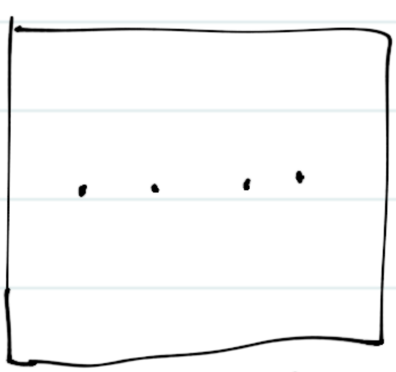


W gen by reflections s_i

$\perp$ to simple roots

<u>Braid gp</u>:

$$\begin{tikzcd} B_W \arrow[r, two heads] \arrow[d, equal] & W \arrow[d, equal] \\ B_n & \Sigma_n \end{tikzcd} \qquad \pi_1(\mathfrak{h}^{reg}/W) \ldots$$



$\mathbb{R}^2$

n pts.

$C_n(\mathbb{R}^2) =$ conf. of n (unlabelled) pts in $\mathbb{R}^2$

$c_0 \in C_n(\mathbb{R}^2)$

$B_n = \pi_1(C_n(\mathbb{R}^2), c_0)$

$$B_W = \langle \hat{s}_i \quad i=1,\dots,n-1 \mid \hat{s}_i\hat{s}_j = \hat{s}_j\hat{s}_i \rangle$$

i, j not nbds.

$(\alpha_i \perp \alpha_j)$

$$\tilde{s}_i \hat{s}_{i+1} \hat{s}_i = \tilde{s}_{i+1} \tilde{s}_i \hat{s}_{i+1}$$

(α_i, α_{i+1} are $120°$)

$$W = \langle t\ \hat{s}_i^2 = 1 \rangle$$

rep. of W, amb. up to T

$T \subseteq B$

<u>Bruhat decomp</u>: $G = \coprod_{w \in W} B\hat{w}B$

row reduced form in linear algebra

<u>Flag variety</u>

$X = \{$ module of Borels $\subset G \}$

$\simeq G/B$ sm. proj.)

$= \{ \{0\} = E_0 \subset E_1 \subset \cdots \subset E_n = \mathbb{C}^n \}$

$\dim E_i = i$

diffeo.

$\simeq G_c / T_c$

$G_c \subset G$ max. compact

$G_c = SU(n)$, $G = SL(n)$

$T_c = G_c \cap T$

Schubert cells

$$X_w = \frac{B\hat{w}B}{B} \subset X$$

$w \in W$

B-orbits $\simeq$ N-orbits on $X^{w \in W}$.

$l : W \longrightarrow \mathbb{N}$ length of

$l(w) =$ min number of s_i needed to express it

Fact 1) $X_w \simeq \mathbb{A}^{l(w)}$

2) $X_{w_1} \subset \overline{X_{w_2}} \Leftrightarrow \underset{\text{min path}}{w_1 \overset{\exists}{\rightsquigarrow} w_2}$ (incr. l)

<u>Cayley graph</u> $(W, S) \rightsquigarrow \Gamma$

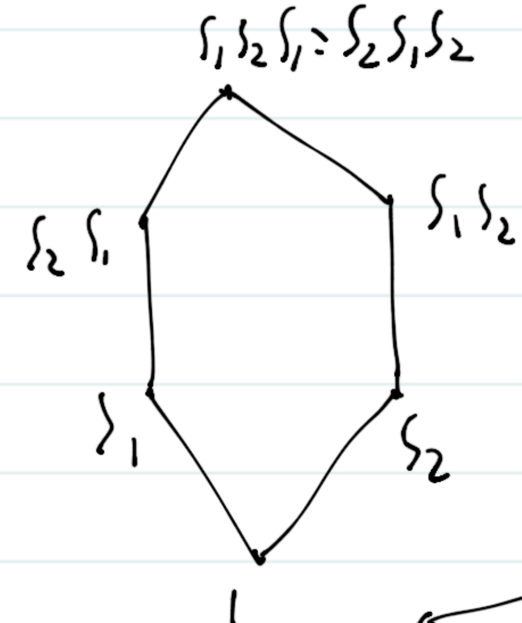
vertices W

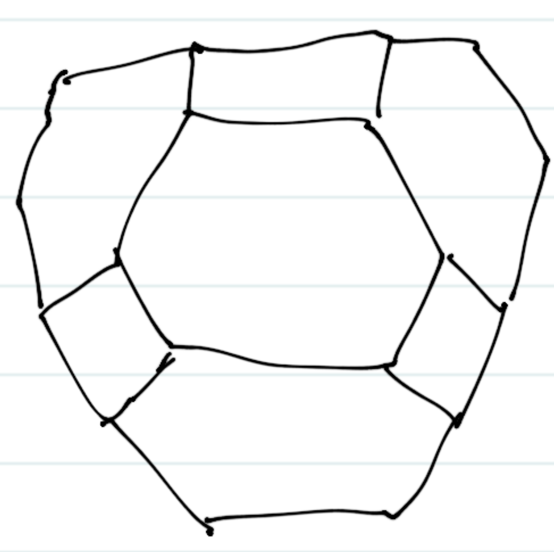
edges: appl. of S

W: Weyl gp.

S: simple gene.

<u>Ex.</u>





Note $\overline{X_w}$ Schubert variety

usually singular.

classifying stack

stack

Convolution $B \backslash G / B \simeq {}_G \backslash\!\!\backslash \left(G/B \times G/B \right) = BB \times_{BG} BB$

The favorite B

$$E_\cdot \longmapsto (E_\cdot^{st}, E_\cdot)$$

Bruhat decomp. $\longleftrightarrow$ rel. pos. decomp.

multiplication $\longleftrightarrow$ convolution

(coming from G)

$\hat{Y}_1, \hat{Y}_2 \subset G$

$Y_1, Y_2 \subset {}_B\backslash G/_B$

$Y_1 * Y_2 \subset {}_B\backslash G/_B$

$\hat{Y}_1 \cdot \hat{Y}_2 \subset G$

multiplication