17 Jan ~~2016~~ 2017 MATH 274 Lecture 1 ①

What I have in mind: What is the Langlands dual grp?

There is a natural geometric answer, but we'll also discuss many other topics along the way.

Today: What is the Langlands dual Lie grp/algebra? (in complex simple case)

Throughout the class, we'll mostly work in the case $k = \bar{k} = \mathbb{C}$.

Defn A (complex) Lie grp is a group object in (complex) manifolds.

This is the data of a manifold G; maps $G \times G \xrightarrow{m} G$, $G \xrightarrow{i} G$, $pt \xrightarrow{e} G$ satisfying some properties (associativity, unit, ...)

When it comes to Langlands duality, we'll want to study things for algebraic grps.

Defn A linear/affine algebraic group G is a grp object in affine varieties.

NB Linear algebraic groups are all also complex Lie groups.

Ex 0) $\mathbb{G}_a = \mathbb{A}^1$ additive grp, $\mathbb{G}_m = GL(1)$ multiplicative group

1) $GL(n)$ = automorphisms of n-dim v.s., $SL(n) = \mathrm{Ker}(GL(n) \xrightarrow{\det} \mathbb{G}_m)$,

$SO(m)$ = matrices preserving inner product, w/ $\det = 1$. (Exercise: Over $\mathbb{C}$, there's a unique nondegenerate quadratic form)

$Sp(2n)$ = matrices preserving nondegenerate skew form (Again there is only one of these $/\mathbb{C}$)

2) $P = \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \text{ which are invertible} \right\}$. This generalizes $\mathbb{G}_a$ (in the upper corner) and $\mathbb{G}_m$ (diagonal) (ie we mean $\mathbb{G}_a = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right\}$, $\mathbb{G}_m = \{ (z) \}$.

Nonexample E elliptic curve is not an affine variety. This is a complex Lie grp which is not an affine algebraic group.

Exercise: If G is a connected compact complex Lie grp, then G is abelian; moreover, $G \cong \mathbb{C}^n / \Lambda$ lattice

Defn A representation of G is a homomorphism $r: G \to GL(V)$

(V will usually be finite-dimensional in this course.)

The repn is called faithful if r is injective.

Exercise: If G is an affine algebraic grp, then there exists a faithful fin. dim. rep.

(Starting point: regular representation $G \to GL(\mathcal{O}(G))$ by left (or right) translation.

Hint: $\mathcal{O}(G)$ is finitely generated.)

Defn A Lie algebra $\mathfrak{g}$ is a vector space with skew-bilinear bracket $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, $(x, y) \mapsto [x, y]$

satisfying the Jacobi identity: $[x, -]: \mathfrak{g} \to \mathfrak{g}$ is a derivation.

17 Jan 2017

In other words, the Jacobi identity is $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$ ("Leibniz")

Adjunction: $\{\text{Lie algebras}\} \underset{\text{forget}}{\overset{U\text{-enveloping}}{\rightleftarrows}} \{\text{assoc. algebras}\}$. That they are adjoint means that we have $\mathrm{Hom}_{Lie}(\mathfrak{g}, A) \cong \mathrm{Hom}_{Assoc}(U\mathfrak{g}, A)$

Explicitly, $U\mathfrak{g} = T\mathfrak{g} / \sim$ ← given by bracket

(So you can embed Lie algebra into associative algebra if you get confused.)

Construction: G Lie group $\rightsquigarrow \mathfrak{g}: T_eG$ Lie algebra.

1) $G \xrightarrow{Ad} GL(\mathfrak{g})$ adjoint rep.

For $Y \in T_eG$, take a path $\gamma_Y(t)$ w/ tangent vector Y at e.

Then we define $Ad_g(Y) := \frac{d}{dt}\big|_{t=0} g\gamma_Y(t)g^{-1}$

2) $\mathfrak{g} \xrightarrow{ad} \mathfrak{gl}(\mathfrak{g})$ is defined by $ad_X(Y) := \frac{d}{ds}\big|_{s=0} Ad_{\gamma_X(s)}(Y)$.

Then we can define the bracket on $\mathfrak{g}$ by $[X,Y] := ad_X(Y)$.

NB: This construction is functorial; in particular, normal subgroup $\rightsquigarrow$ Lie ideal.

Exercise: If G is connected, $\mathrm{Hom}_{Lie\ grps}(G,H) \hookrightarrow \mathrm{Hom}_{Lie\ algs}(\mathfrak{g},\mathfrak{h})$ is an injection.

(Think of Lie algebra as telling you about the formal group. If G is connected, then induced morphisms of formal groups can be used to reconstruct morphisms of Lie groups.)

Ex 0) $\mathrm{Lie}(\mathbb{G}_a) = \mathrm{Lie}(\mathbb{G}_m) = \mathbb{A}^1$ w/ trivial bracket.

1) $\mathfrak{gl}(n)$, $\mathfrak{sl}(n)$, $\mathfrak{so}(n)$, $\mathfrak{sp}(2n)$

2) $\mathfrak{p} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ (but with no constraction of invertibility)

Defn 1) An affine algebraic group G is reductive if the category $\mathrm{Rep}(G)$ of fin-dim algebraic repns is semisimple. (i.e., every simple object is a direct sum of simple objects).

2) A Lie algebra $\mathfrak{g}$ is reductive if the adjoint representation is semisimple. There are other equivalent formulations of this concept.

Defn. • The radical of G is the (identity component) of its maximal normal solvable subgroup.

(S is solvable if it admits a filtration $S \supset S_1 \supset S_2 \cdots \supset S_k = \{1\}$ whose quotients are abelian)

• The radical of $\mathfrak{g}$ is its maximal solvable ideal.

Defn: G affine algebraic group or $\mathfrak{g}$ Lie algebra is called semisimple if its radical is trivial.

Exercise $\mathfrak{g}$ is reductive $\iff \mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{z}$ (semisimple Lie alg; center)

Defn The unipotent radical of an affine algebraic grp G is the subgroup $Rad^u(G)$ of the radical $Rad(G)$ consisting of unipotent elements.
($g \in G$ is unipotent if it acts with unipotent Jordan form on any finite-dim. subrepresentation $V \subset \mathcal{O}(G)$)

Exercise An affine algebraic grp G is reductive $\iff Rad^u(G)$ is trivial.
(From now on in this course we'll only study affine algebraic grps.)

NB For reductive Lie algebras, we have $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{z}$; for reductive Lie grps, the analogous sequence might not split: in general we have $1 \to Z \to G \to G_{ss} \to 1$

Caution 1) G reductive $\not\Rightarrow G = Z \times G_{ss}$ (see note above)
2) G reductive $\Rightarrow \mathfrak{g}$ reductive, but $\mathfrak{g} = Lie(G)$ reductive $\not\Rightarrow G$ reductive.

For geom/physicists: Why are we interested in reductive grps? For G reductive, we can always choose a (unique up to conjugation) maximal compact subgroup $G_c \subset G$, and the classification of red. alg groups is in bijection with the classification of compact Lie groups.
But the payoff is that now we have algebraic geometry as a tool.

Next time: We'll recall the classification of simple Lie algebras, and we'll explain what Langlands duality does to this list.