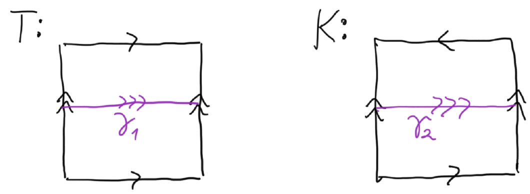
MATH 215A FALL 2020 MIDTERM 2 SOLUTIONS

Exercise 1. Let $T = S^1 \times S^1$ be the torus and let K be the Klein bottle. Consider embeddings $\gamma_1 : S^1 \to T$ and $\gamma_2 : S^1 \to K$ whose images are the oriented circles depicted in the following picture:



Let $X = T \bigcup_{S^1} K$ be the space obtained from the disjoint union of T and K by identifying the points $\gamma_1(t)$ and $\gamma_2(t)$ for each t in S^1. Compute the homology groups of X.

Solution. Note that T and K are subspaces of X, and their intersection inside X is homeomorphic to S^1. Choose open neighborhoods U of T and V of K such that the inclusions $T \to U$, $K \to V$ and $S^1 \to U \cap V$ are homotopy equivalences. Recall that we have

$$H_n(T) = \begin{cases} 0 & \text{if } n > 2 \\ \mathbb{Z} & \text{if } n = 2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 0 \end{cases} \qquad H_n(K) = \begin{cases} 0 & \text{if } n > 1 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 0 \end{cases} \qquad H_n(S^1) = \begin{cases} 0 & \text{if } n > 1 \\ \mathbb{Z} & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 0 \end{cases}$$

and therefore similar values also for the homology of U, V and $U \cap V$.

We apply Van Kampen for the decomposition $X = U \cup V$. For each $n > 2$ we obtain an exact sequence

$$H_n(U) \oplus H_n(V) \to H_n(X) \to H_{n-1}(U \cap V)$$

which simplifies to $0 \to H_n(X) \to 0$. It follows that $H_n(X) = 0$ for $n > 2$.

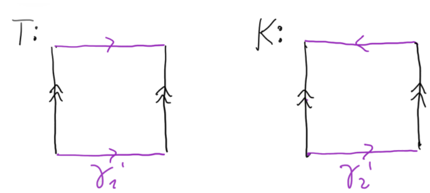
We also have an exact sequence

$$H_2(U \cap V) \to H_2(U) \oplus H_2(V) \to H_2(X) \to H_1(U \cap V) \to H_1(U) \oplus H_1(V) \to H_1(X) \to \widetilde{H}_0(U \cap V)$$

which simplifies to

$$0 \to \mathbb{Z} \to H_2(X) \to \mathbb{Z} \xrightarrow{(j_1, j_2)} (\mathbb{Z} \oplus \mathbb{Z}) \oplus (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \to H_1(X) \to 0.$$

We note that the morphisms j_1, j_2 are the morphisms induced on H_1 by γ_1 and γ_2. These maps are in turn homotopic to embeddings γ_1', γ_2' whose images are the oriented circles depicted in the following picture:



It follows that we have

$$j_1(1) = (0,1) \in \mathbb{Z} \oplus \mathbb{Z} \qquad\qquad j_2(1) = (0,1) \in \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Therefore the map (j_1, j_2) is injective, and our previous exact sequence breaks up into two exact sequences

$$0 \to \mathbb{Z} \to H_2(X) \to 0$$

and

$$0 \to \mathbb{Z} \xrightarrow{(0,1,0,1)} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to H_1(X) \to 0.$$

We see from our first sequence that $H_2(X) = \mathbb{Z}$. The second sequence admits a splitting given by the map $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ of projection onto the second coordinate. We conclude that $H_1(X)$ is isomorphic to the kernel of this projection, which is $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

We finally observe that X is path connected and therefore $H_0(X) = \mathbb{Z}$. In conclusion, we have

$$H_n(X) = \begin{cases} 0 & \text{if } n > 2 \\ \mathbb{Z} & \text{if } n = 2 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 0. \end{cases}$$

Exercise 2. Let $n \geq 2$ and consider the standard embedding $i : \mathbb{RP}^1 \to \mathbb{RP}^n$, induced by passing to the quotient the map $(\mathbb{R}^2 - 0) \to (\mathbb{R}^{n+1} - 0)$ which sends (x_1, x_2) to $(x_1, x_2, 0, 0, \dots, 0)$.

(a) Show that if n is odd then there exists a neighborhood U of $i(\mathbb{RP}^1)$ inside $\mathbb{RP}^n$ and a homeomorphism $h : U \to \mathbb{RP}^1 \times \mathbb{R}^{n-1}$ such that for every p in $\mathbb{RP}^1$ we have $hi(p) = (p, 0)$.

(b) Show that if a pair (U, h) as in (a) exists, then n is odd.

Solution. **(a)** Consider the subspace H inside $\mathbb{R}^{n+1}$ consisting of those points $(x_1, x_2, \dots, x_{n+1})$ such that (x_1, x_2) has Euclidean norm 1, and $x_1 > 0$. Note that H is homeomorphic to $[0, 1] \times \mathbb{R}^{n-1}$. The projection $H \to \mathbb{RP}^n$ factors through the space H' obtained from H by identifying $(0, p)$ with $(1, -p)$ for each p in $\mathbb{R}^{n-1}$. The resulting map $j : H' \to \mathbb{RP}^n$ is an open embedding whose image is an open neighborhood of $i(\mathbb{RP}^1)$. Moreover, the inclusion $\mathbb{RP}^1 \to H'$ induced by corestriction of i is the same as the map obtained by passage to the quotient of the composite map

$$[0, 1] \to [0, 1] \times \mathbb{R}^{n-1} = H \to H'$$

where the first map sends t to $(t, 0)$, and the second map is the canonical projection.

Let $\gamma : [0, 1] \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be a homotopy from the identity $\mathrm{id}_{\mathbb{R}^{n-1}}$ to $-\mathrm{id}_{\mathbb{R}^{n-1}}$, such that for each t in $[0, 1]$ the induced map $\gamma_t : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is an invertible linear transformation (this exists since $n - 1$ is even). Then we have a homeomorphism

$$(\mathrm{id}_{[0,1]}, \gamma) : [0, 1] \times \mathbb{R}^{n-1} \to [0, 1] \times \mathbb{R}^{n-1}.$$

This induces a homeomorphism between H' and the quotient of $[0, 1] \times \mathbb{R}^{n-1}$ by the equivalence relation which identifies $(0, p)$ with $(1, p)$ for each p in $\mathbb{R}^{n-1}$. The latter is in turn homeomorphic to $\mathbb{RP}^1 \times \mathbb{R}^{n-1}$. The resulting homeomorphism between $U = j(H')$ and $\mathbb{RP}^1 \times \mathbb{R}^{n-1}$ satisfies the desired condition.

(b) Assume that n is even. We will compare the homology of U relative to the complement of $i(\mathbb{RP}^1)$ with the the homology of $\mathbb{RP}^1 \times \mathbb{R}^{n-1}$ relative to the complement of $\mathbb{RP}^1 \times \{0\}$, and show that these are different.

We begin with $H_\bullet(U, U - i(\mathbb{RP}^1))$. By excision, this is the same as $H_\bullet(\mathbb{RP}^n, \mathbb{RP}^n - i(\mathbb{RP}^1))$. Note that $\mathbb{RP}^n - i(\mathbb{RP}^1)$ deformation retracts to the complement of a point inside $\mathbb{RP}^{n-1}$, which in turn deformation retracts to a copy of $\mathbb{RP}^{n-2}$. Here the embedding $g : \mathbb{RP}^{n-2} \to \mathbb{RP}^n$ is obtained passing to the quotient the map $(\mathbb{R}^{n-1} - 0) \to (\mathbb{R}^{n+1} - 0)$ which sends $(x_1, x_2, \dots, x_{n-1})$ to $(0, 0, x_1, x_2, \dots, x_{n-1})$.

Observe that after choosing an appropriate cell structure on $\mathbb{RP}^n$, we can think about g as the inclusion of the $(n - 2)$-skeleton (the cell structure being obtained as the usual one, by reversing the order of the coordinates on $\mathbb{RP}^n$). Hence g induces an isomorphism in homology in all degrees except in degree $n - 1$, where g induces the zero map between $H_{n-1}(\mathbb{RP}^{n-2}) = 0$ and $H_{n-1}(\mathbb{RP}^n) = \mathbb{Z}/2\mathbb{Z}$ (here we are using the fact that n is even). It now follows from the long exact sequence in homology for the pair $(\mathbb{RP}^n, g(\mathbb{RP}^{n-2}))$ together with the fact that the inclusion of $g(\mathbb{RP}^{n-2})$ inside $\mathbb{RP}^n - i(\mathbb{RP}^1)$ is a homotopy equivalence, that we have isomorphisms

$$H_{n-1}(\mathbb{RP}^n, \mathbb{RP}^n - i(\mathbb{RP}^1)) = H_{n-1}(\mathbb{RP}^n, g(\mathbb{RP}^{n-2})) = \mathbb{Z}/2\mathbb{Z}$$

and that every other relative homology group is zero.

We now consider the homology of $\mathbb{RP}^1 \times \mathbb{R}^{n-1}$ relative to $\mathbb{RP}^1 \times \mathbb{R}^{n-1} - \mathbb{RP}^1 \times \{0\}$. Observe that the space $\mathbb{RP}^1 \times \mathbb{R}^{n-1} - \mathbb{RP}^1 \times \{0\}$ deformation retracts to a copy of $\mathbb{RP}^1 \times S^{n-2}$. The homology of this can be computed via Mayer-Vietoris in various ways. For instance, it can be computed using example 2.48 in Hatcher, which leads to short exact sequences

$$0 \to H_k(S^{n-2}) \to H_k(\mathbb{R}\mathrm{P}^1 \times S^{n-2}) \to H_{k-1}(S^{n-2}) \to 0$$

for every $k \geq 1$. This implies in particular that $H_{n-1}(\mathbb{R}\mathrm{P}^1 \times S^{n-2})$ is isomorphic to $\mathbb{Z}$ if $n > 2$, or $\mathbb{Z} \oplus \mathbb{Z}$ if $n = 2$. Meanwhile, $\mathbb{R}\mathrm{P}^1 \times \mathbb{R}^{n-1}$ deformation retracts to a copy of $\mathbb{R}\mathrm{P}^1$, so it has the homology of a circle.

It now follows from the long exact sequence in homology for the pair $(\mathbb{R}\mathrm{P}^1 \times \mathbb{R}^{n-1}, \mathbb{R}\mathrm{P}^1 \times S^{n-2})$ together with the fact that the inclusion of $\mathbb{R}\mathrm{P}^1 \times S^{n-2}$ inside $\mathbb{R}\mathrm{P}^1 \times \mathbb{R}^{n-1} - \mathbb{R}\mathrm{P}^1 \times \{0\}$ is a homotopy equivalence, that we have an exact sequence

$$H_n(\mathbb{R}\mathrm{P}^1 \times \mathbb{R}^{n-1}, \mathbb{R}\mathrm{P}^1 \times \mathbb{R}^{n-1} - \mathbb{R}\mathrm{P}^1 \times \{0\}) \to \mathbb{Z} \to 0$$

when $n > 2$, or

$$H_n(\mathbb{R}\mathrm{P}^1 \times \mathbb{R}^{n-1}, \mathbb{R}\mathrm{P}^1 \times \mathbb{R}^{n-1} - \mathbb{R}\mathrm{P}^1 \times \{0\}) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$$

when $n = 2$. It follows in particular that $H_n(\mathbb{R}\mathrm{P}^1 \times \mathbb{R}^{n-1}, \mathbb{R}\mathrm{P}^1 \times \mathbb{R}^{n-1} - \mathbb{R}\mathrm{P}^1 \times \{0\})$ is nonzero. However we have also shown that when n is even the relative homology group $H_n(U, U - i(\mathbb{R}\mathrm{P}^1))$ is zero. This is a contradiction, so we must have that n is odd.

Remark. With a little more care, one can show that the homology of $\mathbb{R}\mathrm{P}^1 \times \mathbb{R}^{n-1}$ relative to $\mathbb{R}\mathrm{P}^1 \times \mathbb{R}^{n-1} - \mathbb{R}\mathrm{P}^1 \times \{0\}$ is $\mathbb{Z}$ in degrees n and $n-1$, and zero everywhere else. Meanwhile, we also showed that the homology of U relative to $U - i(\mathbb{R}\mathrm{P}^1)$ is $\mathbb{Z}/2\mathbb{Z}$ in degree $n-1$, and zero everywhere else. An instructive exercise would be to repeat these computations working with $\mathbb{Z}/2\mathbb{Z}$ coefficients. In this case, both relative homologies will look identical, with copies of $\mathbb{Z}/2\mathbb{Z}$ in degrees n and $n-1$. One way in which one can phrase the outcome of this computation is that the inclusion i admits a relative orientation integrally if and only if n is odd, and it always admits a relative orientation with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

Exercise 3. Let X, Y be path connected, locally path connected, and semilocally simply connected topological spaces. Denote by $\mathrm{Cov}(X)$ (resp. $\mathrm{Cov}(Y)$) the collection of isomorphism classes of (not necessarily path connected) covering spaces of X (resp. Y). Let $f : X \to Y$ be a continuous map, and consider the function $f^* : \mathrm{Cov}(Y) \to \mathrm{Cov}(X)$ which sends the isomorphism class of a covering space $p : E \to Y$ to the isomorphism class of its base change $p' : E \times_Y X \to X$. Show that f^* is a bijection if and only if f induces an isomorphism between the fundamental groups of X and Y.

First solution. One way to solve this problem is to use the relation between covering spaces and group actions from page 68 in Hatcher.

Pick a basepoint x_0 in X and let $y_0 = f(x_0)$. We then have that $\mathrm{Cov}(X)$ (resp. $\mathrm{Cov}(Y)$) can be equivalently described as the collection $\mathrm{Set}^{\pi_1(X,x_0)}$ (resp. $\mathrm{Set}^{\pi_1(Y,y_0)}$) of isomorphism classes of sets with a left action of $\pi_1(X, x_0)$ (resp. $\pi_1(Y, y_0)$). Given a covering space $p : E \to Y$, one attaches to it the set $p^{-1}(y_0)$. This is equipped with the $\pi_1(Y, y_0)$-action defined so that for each class $[\gamma]$ in $\pi_1(Y, y_0)$ and element e in $p^{-1}(y_0)$ we have

$$[\gamma] \cdot e = \widetilde{\gamma}(0)$$

where $\widetilde{\gamma}$ is the unique lift of γ to a path in E such that $\widetilde{\gamma}(1) = e$.

Consider now the covering space $p' : E \times_Y X \to X$. We have a bijection of sets $p'^{-1}(x_0) = p^{-1}(y_0)$ induced from the projection $E \times_Y X \to E$. Let e be an element of this set, and let $[\mu]$ be a class in $\pi_1(X, x_0)$. Let $\widetilde{\mu}$ be the lift of μ to $E \times_Y X$ such that $\widetilde{\mu}(1) = e$. Then the image of $\widetilde{\mu}$ under the projection $E \times_Y X \to E$ is a lift of $f\mu$ with endpoint e. It follows that

$$[\mu] \cdot e = [f\mu] \cdot e$$

where on the left we are using the action of $\pi_1(X, x_0)$, and on the left the action of $\pi_1(Y, y_0)$. We conclude that the map

$$f^* : \mathrm{Cov}(Y) \to \mathrm{Cov}(X)$$

is equivalent to the map

$$\mathrm{Res}_{f_*} : \mathrm{Set}^{\pi_1(Y,y_0)} \to \mathrm{Set}^{\pi_1(X,x_0)}$$

that sends the isomorphism class of a set with $\pi_1(Y, y_0)$-action S to the class of S with the action of $\pi_1(X, x_0)$ induced by restriction along $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

If f induces an isomorphism on fundamental groups then Res_{f_*} is also a bijection, and therefore f^* is also a bijection. Conversely, assume that f^* (and therefore Res_{f_*}) is a bijection. Since the image of Res_{f_*} has to contain the class of a set with a free $\pi_1(X, x_0)$-action, we have that $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is necessarily injective.

It remains to show that f_* is surjective. Assume for the sake of contradiction that this is not the case, and consider the set $S = \pi_1(Y, y_0)/ \mathrm{Im}(f_*)$ equipped with its natural $\pi_1(Y, y_0)$-action. Then $\mathrm{Res}_{f_*}(S)$ admits a fixed point, and in particular it can be written as the disjoint union of two nonempty sets with $\pi_1(X, x_0)$-action. Since Res_{f_*} is a bijection and preserves disjoint unions we conclude that S itself can be written as the union of two nonempty sets with $\pi_1(Y, y_0)$-action, which contradicts the fact that the action of $\pi_1(Y, y_0)$ on S is transitive. Hence f_* is surjective and an isomorphism, as desired.

Second solution. Another way to solve this problem is to use the classification of path connected covering spaces from theorem 1.38.

Pick a basepoint x_0 in X and let $y_0 = f(x_0)$. Assume first that $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism. Any covering space is a disjoint union of connected covering spaces, and pullbacks of covering spaces map disjoint unions to disjoint unions. To show that f^* is a bijection, it suffices to show that f^* maps isomorphism classes of connected covering spaces to isomorphism classes of connected covering spaces, and that it in fact induces a bijection between those.

Let $p : E \to Y$ be a connected covering space. Since X is path connected, to show that $E \times_Y X$ is path connected it suffices to show that for every pair of elements e, e' in $p'^{-1}(x_0)$ there is a path in $E \times_Y X$ from e to e'. Let $\widetilde{\gamma}$ be a path in E from e to e' (where we think about these now as elements in $p^{-1}(y_0)$). Then $p(\widetilde{\gamma})$ is a loop based at y_0. Let μ be a loop based at x_0 such that $f_*[\mu] = [p\widetilde{\gamma}]$, and let $\widetilde{f\mu}$ be the lift of $f\mu$ to E such that $\widetilde{f\mu}(0) = e$. Since $f\mu$ is homotopic to $p(\widetilde{\gamma})$ as based loops, we have that $\widetilde{f\mu}(1) = \widetilde{\gamma}(1) = e'$. The pair $(\mu, \widetilde{f\mu})$ defines then a path in $E \times_Y X$ from e to e'.

We now show that f^* induces a bijection between isomorphism classes of connected covering spaces on X and Y. Let $p : E \to Y$ be a connected covering space and let e be an element in $p^{-1}(y_0) = p'^{-1}(x_0)$. Let μ be a loop in X based at x_0, and let $\widetilde{\mu}$ be its lift to $E \times_Y X$ such that $\widetilde{\mu}(0) = e$. Let $\widetilde{f\mu}$ be the lift of $f\mu$ to E such that $\widetilde{f\mu}(0) = e$. Then $\widetilde{f\mu}$ is the composition of $\widetilde{\mu}$ with the projection $E \times_Y X \to E$. We conclude that $\widetilde{f\mu}$ is a loop in $E \times_Y X$ if and only if $\widetilde{\mu}$ is a loop in Y. This means that $p'_*\pi_1(E \times_Y X, e)$ is the preimage of $p_*\pi_1(E, e)$ under f_*. Our claim now follows from the fact that f_* is a group isomorphism, together with the classification theorem for path-connected covering spaces.

Conversely, assume that f^* is a bijection. We have to show that $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism. Since f^* maps disjoint unions to disjoint unions, we have that f^* restricts to a bijection on isomorphism classes of connected covering spaces. As discussed in the previous paragraph, for each connected covering space $p : E \to Y$ and element e in $p^{-1}(y_0)$ we have that $p'_*\pi_1(E \times_Y X, e)$ is the preimage under f_* of $p_*\pi_1(E, e)$. Since the latter always contains the kernel of f_*, we conclude that f_* is necessarily injective, otherwise the class of the fundamental cover of X would not be in the image of f^*. Furthermore, since the preimages under f_* of $\mathrm{Im}(f_*)$ and $\pi_1(Y, y_0)$ agree, we have that the conjugacy classes of $\mathrm{Im}(f_*)$ and $\pi_1(Y, y_0)$ are the same. Since $\pi_1(Y, y_0)$ is normal inside itself, we conclude that $\mathrm{Im}(f_*) = \pi_1(Y, y_0)$, and therefore f_* is surjective.