215A Lecture 8 (W 9/23/20) Covering Spaces

Def: Covering Space \( p: \tilde{X} \to X \)

such that \( \forall x \in X \exists U \subset X \) nbhd of \( x \)

such that \( p^{-1}(U) = \bigsqcup_{\alpha} U_{\alpha} \to U \)

\[ p|_{U_{\alpha}}: U_{\alpha} \xrightarrow{\cong} U \]

Remark: Some may require:

1) \( p \) surjective

2) \( p_*: \pi_0(\tilde{X}) \xrightarrow{\cong} \pi_0(X) \)

3) Based:

\( (\tilde{X}, \tilde{x}_0) \to (X, x_0) \)...

Example: \( X = S^1 = \mathbb{R}/\mathbb{Z} \) real numbers mod \( \mathbb{Z} \)

\( \hat{X} = S^1 \xrightarrow{p_\theta} S^1 = X \)

\( \theta \mapsto n\theta \)

\( \tilde{X} \)

\( \downarrow \xrightarrow{p} \)

\( \hat{X} = \mathbb{R} \xrightarrow{p} S^1 = X \)

universal cover

All path covers of \( S^1 \) are in above list

(possibly \( n \) and \( -n \) are isomorphic)
2) $X = S' \vee S^1$

To construct coverings $\widetilde{X} \to X$
play lego with pieces

Labels tell you the map $p$.

Ex 1)

$p_* : \pi_1 = \langle a, b, b^{-1}ab \rangle \xrightarrow{\text{subgroup}} \pi_1 = \langle a, b \rangle$

Note: $\text{gim} b^2$ - in fact normal subgroup since covering space exchanging two lifts of $x_0$

Show $\text{Image}(p_*)$ is equal to image in other example.

In general (we'll see): $p_*$ injective

and in fact covering spaces $\overline{\text{space}} = \{ \text{subgroups of } \pi_1 \}$

Furthermore, $\langle \text{based covering spaces} \rangle \! / \! \text{conjugacy} = \{ \text{subgroups of } \pi_1 \}$

$\text{Image}(p_* (x, y)) \to \text{Image}(p_*(x, \overline{x_0}))$
2) \[
\pi_1 = \langle b^n a b^{-n} | n \in \mathbb{Z} \rangle \hookrightarrow \pi_1 = F = \langle a, b \rangle
\]
Free on these many gens!

In general (we'll see): Image (\pi_1) is a normal subgroup.

Assume enough path-connectedness.

3) \[
\pi_1 = \langle a, b, a^{-1} b, b^{-1} a^{-1} b^{-1} \rangle \quad \pi_1 = \langle b^2, b^2, ab a^{-1}, bab^{-1} \rangle
\]

Even show the two \( \pi_1 \)'s are conjugate in \( \pi_1(x, x_0) \cong F^2 \) but not equal.

4) Universal cover

\[
\pi_1 = \langle 1 \rangle
\]
Lifting properties of covering spaces
\[ \widetilde{f}_0 \quad \text{initial condition upstairs} \]
\[ \text{Def } f_t Y \rightarrow Z \text{ homotopy lifting property (h.l.p.)} \]
\[ \tilde{f}_t \quad \text{homotopy upstairs} \]
\[ I \times Y \xrightarrow{f_t} X \quad \text{homotopy downstairs} \]

Prop: \( \tilde{x} \rightarrow x \) covering has h.l.p.
We already proved when proving \( \pi_1(S^1, x_0) = \mathbb{Z} \)

Applications
Prop: \( \tilde{x} \rightarrow x \) covering \( \Rightarrow P^* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0) \) is injective.

Image \((P^*) = \{ \text{loops in } (X, x_0) \text{ lifting to loops in } (\tilde{X}, \tilde{x}_0) \}\) (rather than just paths)

\[ \text{Pf } \text{Suppose } \tilde{g} \in \text{Ker}(P^*) \]
\[ P \tilde{g} \sim \text{triv loop} \]
\[ \text{homo}\text{topic} \quad \text{h.l.p.} \]
\[ \tilde{g} \sim \text{triv loop} \quad \text{homotopy} \]

Conclude: \( \text{Ker}(P^*) = \{1\} \) so \( P^* \) injective.
Prop \( p : \tilde{X} \to X \) covering \( \Rightarrow \left\{ \pi_1(X, x_0) : \pi_1(\tilde{X}, \tilde{x}_0) \right\} \)

\( \tilde{X}, X \) path-connected

**Proof**

\( \Phi : H \to p^{-1}(x_0) \)

Pre-image of \( x_0 \)

Set of \( H \)-coverts \( H_g \)

\( \Phi(\gamma) = \tilde{\gamma}(1) \) where \( \tilde{\gamma} \) is a lift of \( \gamma \) to a path starting at \( \tilde{x}_0 \)

Use prior Prop to see \( \Phi \) well-defined on cosets.

Claim \( \Phi \) bijection

\( \Phi_{\text{surj}} : \tilde{X} \text{ path-connected} \Rightarrow \text{ any } y \in p^{-1}(x_0) \text{ is covered by a path } \tilde{\gamma} : \tilde{x}_0 \to y \)

Check \( \tilde{\gamma} \) lift of \( p\tilde{\gamma} \)

\( \Phi_{\text{inj}} \quad \Phi(\tilde{\gamma}_1) = \Phi(\tilde{\gamma}_2) \Rightarrow \tilde{\gamma}_1, \tilde{\gamma}_2 \text{ lift to loop at } x_0 \)

By prior Prop, \( x_1 \tilde{\gamma}_2 \in H = \text{Image}(p) \) so \( x_1, x_2 \) same coxet.
Prop \( \tilde{\tau} \rightarrow (\hat{X}, \tilde{x}_0) \) \( \downarrow \) \( p \) covering

\( (Y, y_0) \xrightarrow{t} (X, x_0) \)

\( Y \) path-connected and locally path-connected

Lifting existence

\[ \exists f \iff \text{Im}(\tilde{\tau}) \subset \text{Im}(\tilde{p}) \]

In particular, if \( \tilde{\tau}_!(Y, y_0) \in \mathbf{1} \)

so \( f \) always exists

Ex. What goes wrong if

\[ f \text{ collapse in evident way} \]