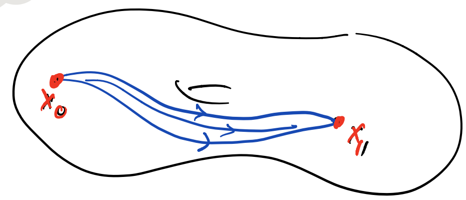
215A Lecture 5 (M 9/14/20) More π_1

Clarification (from last lecture and homework)

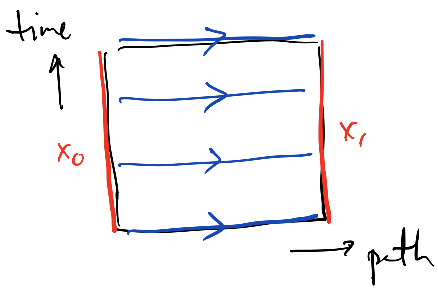
Convention for "homotopy of paths"

fix $x_0, x_1 \in X$

consider $F : \underset{\text{path}}{I} \times \underset{\text{time of homotopy}}{I} \longrightarrow X$



X



Used in proof of $\pi_1(S^1, 1) \simeq \mathbb{Z}$

key properties of $p : \mathbb{R} \longrightarrow S^1$

a) unique path lifting

$$\begin{tikzcd} \{0\} \arrow[r, "\tilde{x}_0"] \arrow[d] & \mathbb{R} \arrow[d, "p"] \\ I \arrow[r, "\gamma"'] \arrow[ur, dashed, "\exists!"] & S^1 \end{tikzcd}$$

b) unique homot. of path lifting

$$\begin{tikzcd} I \times \{0\} \arrow[r, "\tilde{x}_0"] \arrow[d] & \mathbb{R} \arrow[d, "p"] \\ I \times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists!"] & S^1 \end{tikzcd}$$

Functoriality of π_1

$\pi_1 : \mathrm{Top}_* \longrightarrow \text{Groups}$

based top spaces

Objs $(X, x_0) \longmapsto \pi_1(X, x_0) = [(S^1, 1), (X, x_0)]$

Morphs $\left((X, x_0) \xrightarrow{f} (Y, f(x_0))\right) \longmapsto f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$

$[\gamma] \longmapsto [f \circ \gamma]$

Check composition identities...

Rmk Easy to check π_1 factors through $\mathrm{Top}_*/\sim$

based homot cat where based homot maps are equal.

Ex $(A, a_0) \xhookrightarrow{i} (X, x_0)$

in general, can't say anything without more info

1) retraction $r : (X, x_0) \to (A, a_0)$, $r \circ i = id_A$

$\Longrightarrow$ i_* injective, r_* surjective

To prove: use functoriality

2) deformation retraction $r_t : (X, x_0) \to (X, x_0)$

$r_t|_A = id_A$, $r_0 = id_X$

$r_1(X) \subset A$

$\Longrightarrow$ i_*, r_{1*} isoms (in fact inverses)

To prove: use prior remark. + functoriality

Prop $f : (X, x_0) \xrightarrow[\uparrow]{\simeq} (Y, f(x_0))$ based map

homot. equiv
but not nec based
(so homot inverse $g : Y \to X$
and homots $gf \simeq id_X$, $fg \simeq id_Y$
are not nec based.)

$\Longrightarrow f_*$ isom.

Lemma $F_t : X \to Y$ homotopy

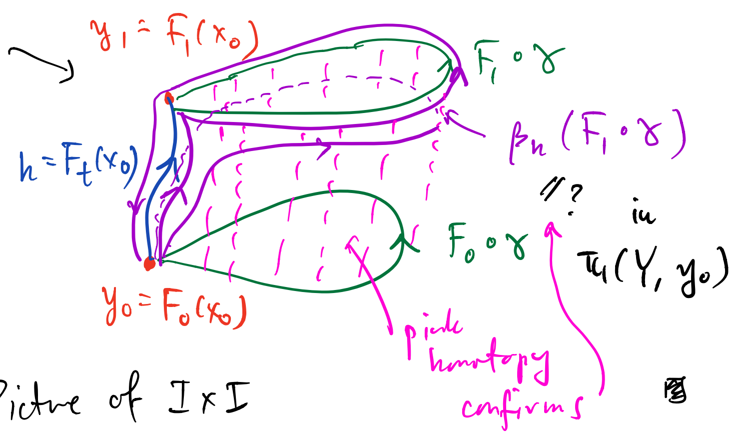
$\Longrightarrow$ we have comm diag

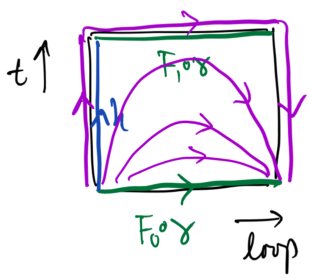
$$\begin{tikzcd} & \pi_1(Y, F_1(x_0)) \arrow[dd, "\simeq", "\beta_h"'] \\ \pi_1(X, x_0) \arrow[ur, "F_{1*}"] \arrow[dr, "F_{0*}"'] \arrow[r, phantom, "\circ"] & \\ & \pi_1(Y, F_0(x_0)) \end{tikzcd}$$

$y_1 = F_1(x_0)$, $F_0(x_0) = y_0$

where $h = F_t(x_0)$
path $y_0 \rightsquigarrow y_1$

Proof. Picture of Y $\rightsquigarrow$





Picture of $I \times I$

∎

<u>Proof</u> of <u>Prop</u>: Let $g : Y \to X$ be homot. inverse

Consider

$$
\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[rr, bend right, "g_* \circ f_*"'] & \pi_1(Y, f(x_0)) \arrow[r, "g_*"] \arrow[rr, bend left] & \pi_1(X, gf(x_0)) \arrow[r, "f_*"] & \pi_1(Y, fgf(x_0))
\end{tikzcd}
$$

composition $g_* \circ f_*$
is an <u>isom</u> by Lemma

<u>Use</u>: F_t homot between $F_0 = gf$, $F_1 = id_X$

Lemma $\Rightarrow$ $g_* f_* = \underbrace{\beta_h}_{\text{isom}} \circ \underbrace{id_{X*}}_{\text{id}}$

$\Rightarrow$ f_* <u>injective</u>

Same argument shows g_* <u>injective</u>

Altogether $g_* f_*$ is an isom, and also composition of injections

$\Rightarrow$ f_* <u>surjection</u> $\blacksquare$

<u>Additional remarks on functoriality.</u>

1) <u>Prop</u>

$$
\begin{tikzcd}
& (X \times Y, (x_0, y_0)) \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
(X, x_0) & & (Y, y_0)
\end{tikzcd}
\quad \text{induces} \quad
\begin{tikzcd}
& \pi_1(X \times Y, (x_0, y_0)) \arrow[dl, "\pi_{X*}"'] \arrow[d, dashed, "\exists! \cong"] \arrow[dr, "\pi_{Y*}"] & \\
\pi_1(X, x_0) \arrow[dr, "\circ"] & \pi_1(X, x_0) \times \pi_1(Y, y_0) \arrow[l] \arrow[r] \arrow[d] & \pi_1(Y, y_0) \arrow[dl, "\circ"'] \\
& \langle 1 \rangle &
\end{tikzcd}
$$

"π_1 takes prod of two based spaces
to prod of their π_1"

2) <u>We'll see</u>: van Kampen $\Rightarrow$ "π_1 takes coprod of two based spaces
to coprod of their π_1"

Aside A domain space, $\begin{tikzcd} X \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$ diagram of spaces

$$\mathrm{Maps}_{Top}\left(A, X \underset{Z}{\times} Y\right) \simeq \mathrm{Maps}_{Diags}\left(\begin{tikzcd} A \arrow[dr, "="'] & & A \arrow[dl, "="] \\ & A & \end{tikzcd}, \begin{tikzcd} X \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd} \right)$$

"maps into fiber prod" — "pairs of maps into X and Y agreeing when mapped to Z"

univ prop of fiber prod.

$$\begin{tikzcd} & A \arrow[dl] \arrow[d, dashed, "\exists!"] \arrow[dr] & \\ X \arrow[dr] & X \underset{Z}{\times} Y \arrow[l] \arrow[r] \arrow[d] & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

Aside to aside $\mathcal{C}$ cat, diag $\begin{tikzcd} X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$

fiber prod (if it exists) is obj $X \underset{Z}{\times} Y$ with maps

$$\begin{tikzcd} & X \underset{Z}{\times} Y \arrow[dl] \arrow[dr] & \\ X \arrow[dr] \arrow[rr, phantom, "\circ"] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

universal in the sense

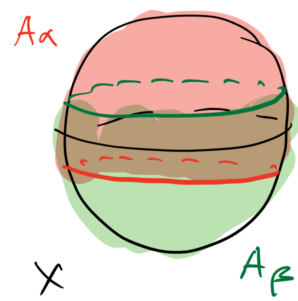
$$\begin{tikzcd} & A \arrow[dl] \arrow[d, dashed, "\exists!"] \arrow[dr] & \\ X \arrow[dr] & X \underset{Z}{\times} Y \arrow[l] \arrow[r] \arrow[d] & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

Back to the topic at hand...

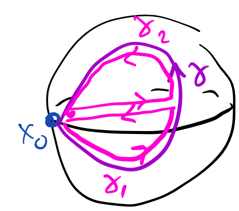
Thm $\pi_1(S^n, x_0) \cong \langle 1 \rangle, \quad n \geq 2.$

Factorization lemma $X = \bigcup_\alpha A_\alpha$

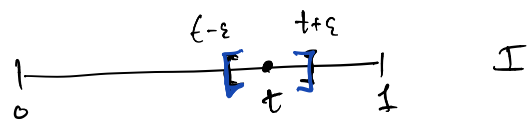
- open, $x_0 \in A_\alpha$
- $A_\alpha \cap A_\beta$ path-conn



$\Rightarrow$ Any $\gamma : (S^1, 1) \to (X, x_0)$
factors up to homot $\gamma \sim \gamma_1 \cdots \gamma_m$
where each $\gamma_i : (S^1, 1) \to (A_{\alpha(i)}, x_0)$



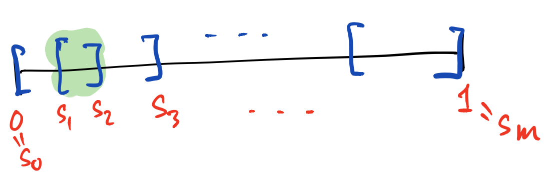
Proof



for each $t \in I$, $\gamma(t) \in$ some A_α,

so $\gamma([t-\varepsilon, t+\varepsilon]) \subset A_\alpha$ for small $\varepsilon = \varepsilon(t)$

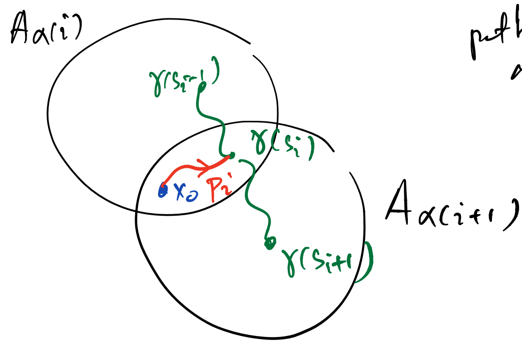
Cover I by such intervals. Since I compact,
can still cover with fin many



Note $\gamma([s_i, s_{i+1}]) \subset$ some $A_{\alpha(i)}$

For each i, choose path $p_i : x_0 \rightsquigarrow \gamma(s_i)$

lying in $\underbrace{A_{\alpha(i)} \cap A_{\alpha(i+1)}}_{\text{path-conn by assumption}}$

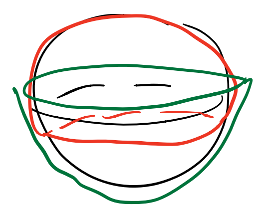


<u>Conclusion</u> $\gamma = (\gamma|_{[\underset{s_0}{\overset{\shortparallel}{0}}, s_1]} \cdot \bar{p}_1)(p_1 \gamma|_{[s_1, s_2]} \bar{p}_2) \cdots$

$(p_{m-1} \gamma|_{[s_{m-1}, \underset{s_m}{\overset{\shortparallel}{1}}]})$

∎

<u>Proof of Thm</u>: Write $S^n = \overset{A_\alpha}{\overset{\shortparallel}{\mathbb{R}^n}} \cup \overset{A_\beta}{\overset{\shortparallel}{\mathbb{R}^n}}$



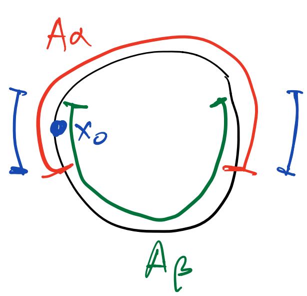
Note $n \geq 2$, $A_\alpha \cap A_\beta$ path conn.

Lemma $\Rightarrow$ factor loops.

But loops in $\mathbb{R}^n$ are all $\sim$ trivial

So all loops in S^n are $\sim$ trivial ∎

<u>$n = 1$</u>



$A_\alpha \cap A_\beta$ not path-conn!