215a Lecture 26 (W 12/2/20) Poincaré duality continued

Last time M compact, R-orientable n-mfd

consistent choice of gen $\mu_x \in H_n(M, M \setminus x; R)$ all $x \in M$

$\Longrightarrow$ $\exists$ fund class $[M] \in H_n(M;R)$

restricts to μ_x, all $x \in M$

Cases
1) $R = \mathbb{Z}/2$: M can. oriented
2) M conn, $\pi_1(M) = \langle 1 \rangle$: M orientable
3) M smooth, $\Lambda^n TM$ trivializable : M orientable

(for ex: M underlying real mfd of a complex mfd)

Thm (Poincaré duality) M compact R-orient. n-mfd with fund class $[M] \in H_n(M, R)$

Then $D_M : H^*(M;R) \xrightarrow{\cong} H_{n-*}(M;R)$

$$D_M(\varphi) = [M] \frown \varphi$$

Cor R = either a field or $\mathbb{Z}$ and we quot by torsion (so U.C.T. : $H^k(M,R) \xrightarrow{\sim} H_k(M,R)^*$)

Then

$$H^k(M,R) \otimes H^{n-k}(M,R) \xrightarrow{\cup} H^n(M,R) \xrightarrow{[M]\cap} H_0(M,R) \xrightarrow{\deg} R$$

is a perfect pairing

Pf. Exercise using $\psi([M] \cap \varphi) = (\varphi \cup \psi)([M])$. ∎

Application $H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[u]/(u^{n+1})$ $\quad |u| = 2$

$H^*(\mathbb{RP}^n;\mathbb{Z}/2) = \mathbb{Z}/2[v]/(v^{n+1})$ $\quad |v| = 1$

Pf. Case of $\mathbb{CP}^n$ ($\mathbb{RP}^n$ is similar).

By induction on n. Consider $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$ linear inclusion

$$H^*(\mathbb{CP}^n) \longrightarrow H^*(\mathbb{CP}^{n-1}) \cong \mathbb{Z}[u]/(u^n) \longleftarrow \text{induction}$$

$u \in H^2(\mathbb{CP}^n) \cong \mathbb{Z}$ generator

By Cor $H^2(\mathbb{CP}^n) \otimes H^{2n-2}(\mathbb{CP}^n) \to \mathbb{Z}$ perfect pairing

u gen $\in H^2(\mathbb{CP}^n)$, u^{n-1} gen $\in H^{2n-2}(\mathbb{CP}^n)$

so $u \cup u^{n-1}$ gen of $H^{2n}(\mathbb{CP}^n)$ ∎

Worthwhile comparison : $H^*(\mathbb{CP}^3)$ vs $H^*(S^2 \times S^4)$

$$H^*(\mathbb{CP}^3) = \mathbb{Z}[u]/(u^4) \qquad |u| = 2$$

$$H^*(S^2\times S^4) = \mathbb{Z}[x,v]/(x^2,v^2) \qquad |x| = 2,\ |v| = 4$$

Now discussion of arguments behind Poincaré duality

Obstacle In comparing "loc const fns" and "loc const dists" we're comparing apples and oranges.

Why?: different functoriality to start...

First key idea Replace "loc const fns" by "compactly-supp loc const fns". These have covariant functoriality under open inclusions (similarly to "loc const dists") !

Cosheaves!

Second key idea : Both can be calculated globally by gluing local calculations (Mayer-Vietoris).

Final basic idea : The two are equal locally! Here we'll use M locally $\simeq \mathbb{R}^n$!

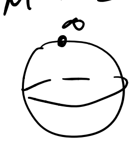
Where do we use M compact?!? "Compactly-supp loc const fns" = "Loc const fns" !

Let's make this more precise

Def. M mfd $\rightsquigarrow$ $\bar{M}$ = 1 pt compactification of M
$= M \cup \infty$

Compactly-supp cohomology

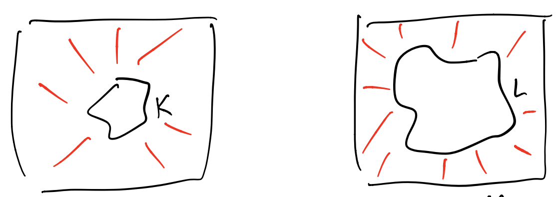
Ex $M = \mathbb{R}^n$
$\rightsquigarrow \bar{M} = S^n$



$$H^*_c(M;G) = H^*(\bar{M}, \infty; G)$$

cohom of pair

Exer 1) $H^*_c(M;G) = \operatorname{colim}_{\substack{K \subset G \\ \text{compact}}} H^*(M, M \setminus K; G)$



M M

$$(M, M\setminus K) \xleftarrow{i} (M, M \setminus L)$$

$$H^*(M, M\setminus K, G) \xrightarrow{i^*} H^*(M, M\setminus L; G)$$

Rmk: H_* also satisfies this! Usual argument that any single cochain has compact image...

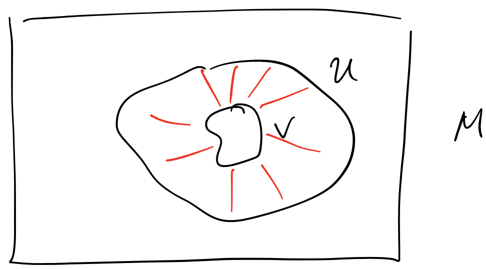
2) Covariant functoriality under open inclusions: $V \underset{\text{open}}{\subset} \mathcal{U} \underset{\text{open}}{\subset} M$

$$H_c(V;G) \longrightarrow H_c(\mathcal{U};G)$$

"wrong way map"

$$\begin{array}{ccc} \cong & & \Vert \\ H^*(\bar V,\infty;G) & & H^*(\bar{\mathcal U},\infty;G) \\ \text{excision} \;\; \cong & \nearrow & \text{induced by map of pairs} \\ & H^*(\bar{\mathcal U},(\mathcal U\setminus V)\cup\infty;G) & \end{array}$$

$$(\bar{\mathcal U},\infty) \longrightarrow (\bar{\mathcal U},(\mathcal U\setminus V)\cup\infty)$$



3) $$H^*_c(\mathbb{R}^n;G) \simeq \begin{cases} G & *=n \\ 0 & \text{else} \end{cases}$$

$$\cong$$

$$H^*(S^n,\infty;G)$$

<u>Relation to homology</u>: generalized cap product

<u>Observe</u>: 1) Construction of fund class $[M]\in H_n(M;R)$ for M compact

generalizes to fund class $[\bar M]\in H_n(\bar M,\infty;R)$

restriction of $[\bar M]$ gens $H_n(M,M\setminus x;R)$ for all $x\in M$.

2) Cap product generalizes to:

$$H^k_c(M;R) \times H_\ell(\bar{M},\infty;R) \longrightarrow H_{\ell-k}(M;R)$$

"comp-supp loc const fns" "loc const dists with arb supp" "loc const dists"

Thm (Poincaré duality without compactness)
M R-orient. n-mfd with fund class $[\bar{M}] \in H_n(\bar{M},\infty;R)$

Then $D_M : H^k_c(M;R) \xrightarrow{\frown} H_{n-k}(M;R)$

$$\varphi \longmapsto [\bar{M}] \frown \varphi$$

Cor Traditional P.D when M compact.

Local-to-Global Lemma $M = U \cup V$ union of opns

Then comm diag of Mayer-Vietoris sequences (up to sign)

$$\begin{tikzcd}
\to H^k_c(U \cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H^k_c(U) \oplus H^k_c(V) \arrow[r] \arrow[d, "D_U \oplus -D_V"] & H^k(U \cup V) \arrow[r] \arrow[d, "D_{U\cup V}"] & \cdots \\
\cdots \to H_{n-k}(U \cap V) \arrow[r] & H_{n-k}(U) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(U \cup V) \arrow[r] & \cdots
\end{tikzcd}$$

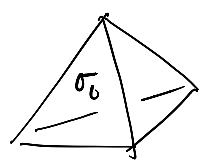
__Local calculation__ $M = \mathbb{R}^n$

__Want isom:__ $H_c^*(\mathbb{R}^n; R) \xrightarrow{[\overline{\mathbb{R}^n}]\frown} H_{n-*}(\mathbb{R}^n; R)$

Both sides are only $\neq 0$ when $*=n$:

$$\begin{tikzcd} H_c^n(\mathbb{R}^n; R) \arrow[r, "{[\overline{\mathbb{R}^n}]\frown}"] \arrow[d, phantom, "\cong" rotate=90] & H_0(\mathbb{R}^n; R) \arrow[d, phantom, "\cong" rotate=90] \\ H^n(S^n; R) \arrow[r, "{[S^n]\frown}"] & H_0(S^n; R) \end{tikzcd}$$

$S^n = \partial \Delta^{n+1}$



Take $\varphi \in H^n(S^n; R)$ gen that is 1 on σ_0 (some face) 0 else

Observe $[S^n] \frown \varphi = \varphi(\sum_i \sigma_i) = \varphi(\sigma_0) = 1$.

So $[S^n]\frown$ in bottom row is an isom.

So $[\overline{\mathbb{R}^n}]\frown$ in top row is as well!

__Rmk__ Proof now involves inductive applications of above facts

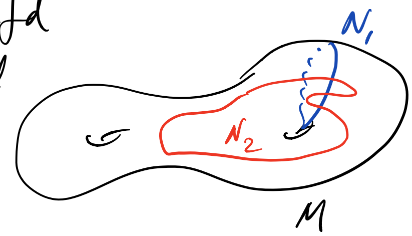
__Nice situation__ (where proof is particularly simple):

suppose we have open cover $\{U_i\}_{i \in I}$ of M such that any intersection $\bigcap_{i \in J} U_i$ is either contractible or empty.

Geometry of cup product via Poincaré duality.

Setup M compact, smooth, oriented n-mfd

$N_1, N_2 \subset M$ compact oriented submanifolds



Then $[N_1] \in H_i(M)$
$[N_2] \in H_j(M)$

Question What is the geometry of

$$D_M\left(D_M^{-1}([N_1]) \cup D_M^{-1}([N_2])\right) \in H_{i+j-n}(M) \;?$$

Answer If N_1, N_2 intersect transversely

then answer is $[N_1 \cap N_2]$ with its induced orientation

(Exer define this)

Ex $M = \mathbb{CP}^n$, $N_1 = \mathbb{CP}^{n-1} = \{[0, z_1, \ldots, z_n]\}$
$N_2 = \mathbb{CP}^{n-1} = \{[z_0, \ldots, z_{n-1}, 0]\}$

$[N_1] = [N_2]$ gen H^2

$N_1 \cap N_2 = \mathbb{CP}^{n-2} = \{[0, z_1, \ldots, z_{n-1}, 0]\}$

$(N_1 \cap N_2)$ sees $H4$

and so on...