Lecture 25 (M 11/30/20) Poincaré Duality

Let's start with a combinatorial form of Poincaré duality we've already discussed (informal discussion):

Setup: $M$ compact nfdl (locally $\approx \mathbb{R}^n$, Hausdorff)

for example $M$ is closed surface.

Suppose $M$ has a triangulation

$$C^*(M, \mathbb{Z}/2) \leftarrow C^*_{n-k}(M, \mathbb{Z}/2)$$

(cochain complex)

chain complex

of dual cells

Consequence

$H^*(M, \mathbb{Z}/2) \rightarrow H_{n-k}(M, \mathbb{Z}/2)$

Poincaré duality!

Question: Can we make this work without reference to any combinatorial topology?
Guideline from analysis: \( H^*(M) = \) "loc. count. fns on \( M \)"
\( H_{\text{loc}}(M) = \) "loc. count. diff's on \( M \)"

Mumford: cup prod. = "prod of fns"

\( H^* \to \text{Hom}(H_*, \mathbb{R}) = \) "pairing of fns and diff's"

If we choose a vol. form \( vol \) on \( M \), then we have map
\[ \text{Functions} \to \text{Distributions} \]
\[ f \mapsto \int_M f(\cdot) \, vol \]

To construct \( H^*(M, \mathbb{R}) \to H_{\text{loc}}(M, \mathbb{R}) \), we need topol. version of "volume form"

Analogous of multiplicity
\[ \text{Fns} \times \text{Diff's} \to \text{Diff's} \]

and we want that multplies cochains with this "volume form".

Remark: When \( \mathbb{R} = \mathbb{Z}/2\mathbb{Z} \), there is a canonical "volume form".
\[ C_{\text{cap product}}: \text{X space, R comm ring, } k \geq l \]

\[ C^k(X, R) \times C^l(X, R) \rightarrow C_{k-l}(X, R) \]

\[ \varphi \times \sigma \rightarrow \sigma \cap \varphi \]

\[ \sigma \cap \varphi = \varphi \left( \sigma \mid s_0, \ldots, s_l \right) \cdot \sigma \mid s_l, \ldots, s_k \]

\[ \text{elt of } R \quad \text{and in } R \text{ chain} \]

\[ 2 \]

\[ \begin{array}{c}
\sigma \\
\downarrow \\
1 \\
\bigtriangleup \\
\sigma \mid s_l, \ldots, s_k \\
\end{array} \]

**Exer 1)** \[ \sigma (\sigma \cap \varphi) = (-1)^l (\varphi \sigma \cap \varphi - \sigma \cap \delta \varphi) \]

so cap prod induces

\[ H^k(X, R) \times H^l(X, R) \rightarrow H^{k-l}(X, R) \]

2) **projection formula:** \( f: X \rightarrow Y \)

\[ f^*(\sigma \cap \varphi) = f^*(\sigma \cap f^*\varphi) \]

3) **interaction with cap prod:** \( \varphi (\sigma \cap \varphi) = (\varphi \cup \varphi)(\sigma) \)
So comm diag
\[
\begin{array}{c}
H^p(X, \mathbb{R}) \rightarrow H_{\mathbb{R}}(H_\mathbb{R}^p(X, \mathbb{R}), \mathbb{R}) \\
\cup \Psi \downarrow \quad \downarrow (\cap \Psi)^* \\
H^{k+t}(X, \mathbb{R}) \rightarrow H_{\mathbb{R}}(H_{k+t}(X, \mathbb{R}), \mathbb{R})
\end{array}
\]

Top version of "rel form"

fundamental class
(or orientation class)

Observe \(M\) n-mfld \(\Rightarrow H_i(M, M \cdot x; \mathbb{R}) \cong \begin{cases} \mathbb{R} & i = n \\
0 & \text{else} \end{cases}\) (homology of sphere around \(x\))

Key question: How canonical is \((*)\)?

When \(R = \mathbb{Z}\): two possible choices since \(\mathbb{Z}\) has two generators \(\pm 1\).

When \(R = \mathbb{Z}/2\): unique choice since \(\mathbb{Z}/2\) has unique 

Def 1) R-orientation of \(M\) at \(x \in M\) is choice of gen 
\(\mu_x \in H_n(M, M \cdot x; \mathbb{R}) \cong \mathbb{R}\)

2) R-orientation of \(M\) is compatible choice of R-orientations at all \(x \in M\).
Meaning of compatible

\[ H_n(M, M \smallsetminus x; \mathbb{R}) \cong H_n(M, M \smallsetminus y; \mathbb{R}) \]

\[ M_x \quad \overleftarrow{\quad} \quad M_y \]

Bundle of orientations

Classical case: \( R = \mathbb{C} \)

\[ \tilde{M} = \left\{ (x, \mu_x) \mid x \in M, \mu_x \text{ group of } H_n(M, M \smallsetminus x) \right\} \]

\[ \pi : \tilde{M} \quad \text{bundle} \quad \mathbb{Z}/2 \quad \text{principal bundle} \]

Exercise: Topologize \( \tilde{M} \) so that \( \pi \) is indeed covering.

"Nearly \((x, \mu_x)\) are compatible"

Exercise: \( M = \mathbb{RP}^2 \implies \tilde{M} = S^2 \)

\[ M = S^2 \implies \tilde{M} = S^1 \sqcup S^2 \]

Observe: \( \tilde{M} \) has a \underline{tautological orientation} \((x, \mu_x) \implies \mu_x\)

Prop Orientation of \( M \) \( \quad \text{section} \quad \Sigma_{M}^\mathcal{F} \)

pf. Exercise.
Cor: \( M \) simply-connected \( \implies \) \( M \) is orientable with exactly \( 2^n \) orientations.

**Remark:** There's an analogous covering space for \( \mathbb{R} \)-orientation.

**Def.** Fund class \( [M] \in H_n(M, \mathbb{R}) \) such that for all \( x \in M \)

\[
H_n(M, \mathbb{R}) \xrightarrow{\sim} H_n(M, M \cdot x, \mathbb{R})
\]

we have \( [M] \) \( \mapsto \) generator (local orientation).

In other words \( [M] \) defines an \( \mathbb{R} \)-orientation of \( M \).

Conversely...

**Theorem:** \( M \) compact conn \( n \)-mfd

1) \( M \) \( \mathbb{R} \)-orientable \( \implies \)

\[
H_n(M, \mathbb{R}) \xrightarrow{\sim} H_n(M, M \cdot x, \mathbb{R}) \quad \text{for all } x \in M
\]

2) \( M \) not \( \mathbb{R} \)-orientable \( \implies \)

\[
H_n(M, \mathbb{R}) \xrightarrow{\text{inj.}} H_n(M, M \cdot x, \mathbb{R}) \supseteq \mathbb{R}
\]

for all \( x \in M \)

with image \( \{ r \in \mathbb{R} \mid 2r = 0 \} \)

3) \( H_i(M, \mathbb{R}) = 0 \quad i > n \)
\( \text{Car } R=\mathbb{Z} \quad H_n(M, \mathbb{Z}) = \begin{cases} \mathbb{Z} & M \text{ nontellable } \\ 0 & M \text{ not nontellable} \end{cases} \)

\( R=\mathbb{Z}/2 \quad H_n(M, \mathbb{Z}/2) = \mathbb{Z}/2 \)

To prove Thur, we'll use:

Let \( M_R = \{ (x, r_x) \mid x \in M, r_x \in H_n(M, M-x, \mathbb{R}) \} \)

not nec gen.

Covering space with fibers \( R \) (in fact a comm ring space /\( M \))

sections = compatible choices of fields in local chart.

Lemma: \( M \) n-mfd, \( A \subset M \) compact

\[ a) \quad \pi | M_R \quad \begin{cases} \alpha \in \text{secin} \quad \Rightarrow \exists! \quad \alpha_A \in H_n(M, M-A) \\ \text{restricting to } \alpha_x \text{ for all } x \in A \quad \text{ via } \quad H_n(M, M-A) \rightarrow H_n(M, M-x) \end{cases} \]

\[ b) \quad H_i(M, M-A; \mathbb{R}) = 0 \quad i > n \]

Lemma \( \Rightarrow \) Thur: \( M \) compact, set \( A = M \)

So \( b) \Rightarrow 3) \)

For 1) and 2), consider natural map

\[ H_n(M, R) \rightarrow \text{sections } (\pi | M_R) \quad a) \Rightarrow \text{ isom} \].
Since \( M \) compact, 1), 2) follow since sections determined by restriction to a point.

We'll leave proof of Lemma to Office Hours or other discussions...

Then (Poincaré duality)

- \( M \) compact, R-oriented \( n \)-mfd
- Let \( \psi_0 \in H_n(M; \mathbb{R}) \) be a fund class

Thus \( \quad H^k(M; \mathbb{R}) \xrightarrow{\sim} H_{n-k}(M; \mathbb{R}) \)

\[ \psi \mapsto [M] \wedge \psi \]