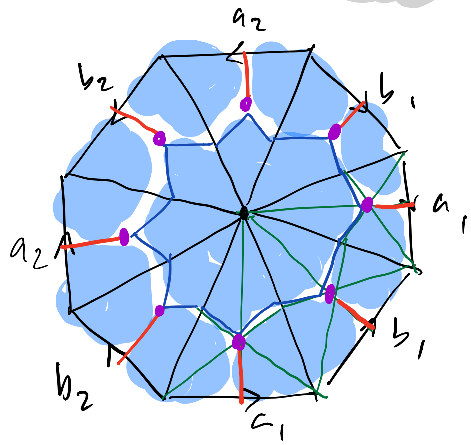
215a Lecture 25 (M 11/30/20) Poincaré Duality

Let's start with a combinatorial form of Poincaré duality we've already discussed (informal discussion):

Setup M compact n-mfd (locally $\simeq \mathbb{R}^n$, Hausdorff)
for example M is closed surface.

Suppose M has a triangulation



$\sigma \rightsquigarrow$ dual cell C_σ
dim $= k$ $\qquad$ dim $= n-k$

$$C^*(M, \mathbb{Z}/2) \simeq C^{\text{dual}}_{n-*}(M, \mathbb{Z}/2)$$

cochain complex of triang. $\qquad$ chain complex of dual cells

Consequence $H^*(M, \mathbb{Z}/2) \xrightarrow{\simeq} H_{n-*}(M, \mathbb{Z}/2)$

Poincaré duality!

Question: Can we make this isom without reference to any comb. str?

Guidance from analysis: $H^*(M)$ = "loc const fns on M"
$H_*(M)$ = "loc const dists on M"

Moreover: cup prod. = "prod of fns"

$H^* \to \mathrm{Hom}(H_*, R)$ = "pairing of fns and dists"

If we choose a vol. form vol on M, then we have map Functions $\to$ Distributions

$$f \longmapsto \int_M (\ \ ) f \cdot \mathrm{vol}$$

measure of integration

To construct $H^*(M,R) \xrightarrow{\sim} H_{n-*}(M,R)$,

we need topol version of "volume form"

and an operation that multiplies cochains with this "volume form".

Analogue of multiplication Funs x Dists $\to$ Dists

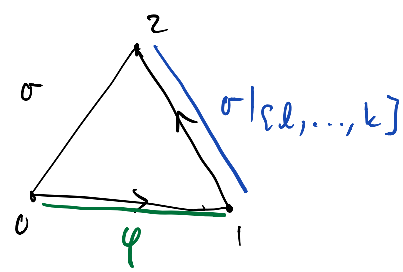
Rmk: When $R = \mathbb{Z}/2\mathbb{Z}$ there is a canonical "volume form"!

Cap product: X space, R comm ring, $k \geq \ell$

$$C^k(X,R) \times C_\ell(X,R) \longrightarrow C_{k-\ell}(X,R)$$
$$\varphi \times \sigma \longmapsto \sigma \frown \varphi$$

$$\sigma \frown \varphi = \underbrace{\varphi(\sigma|_{[0,\dots,\ell]})}_{\text{elt of } R} \cdot \underbrace{\sigma|_{[\ell,\dots,k]}}_{\text{chain}}$$

prod in R



Exer 1) $\partial(\sigma \frown \varphi) = (-1)^\ell (\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$

so cap prod induces

$$H^k(X,R) \times H_\ell(X,R) \longrightarrow H_{k-\ell}(X,R)$$

2) *projection formula:* $f: X \to Y$

$$f_*(\sigma) \frown \varphi = f_*(\sigma \frown f^*\varphi)$$

3) *interaction with cup prod:* $\psi(\sigma \frown \varphi) = (\varphi \smile \psi)(\sigma)$

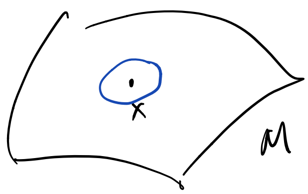
So comm diag

$$\begin{array}{ccc} H^{\ell}(X,R) & \to & \mathrm{Hom}(H_\ell(X,R),R) \\ \downarrow {\scriptstyle \cup\varphi} & & \downarrow {\scriptstyle (\cap\varphi)^*} \\ H^{k+\ell}(X,R) & \to & \mathrm{Hom}(H_{k+\ell}(X,R),R) \end{array}$$

Top version of "vol form" — fundamental class (or orientation class).

Observe M n-mfd $\Rightarrow$ $H_i(M, M\setminus x; R) \overset{(*)}{\simeq} \begin{cases} R & i=n \\ 0 & \text{else} \end{cases}$

(red. homology of sphere around x)



Key question How canonical is $(*)$?

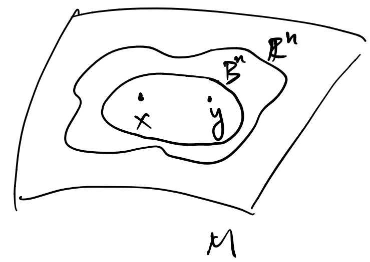
When $R=\mathbb{Z}$: two possible choices since $\mathbb{Z}$ has two generators ± 1

When $R=\mathbb{Z}/2$: unique choice since $\mathbb{Z}/2$ has unique gen 1.

Def: 1) R-orientation of M at $x\in M$ is choice of gen $\mu_x \in H_n(M, M\setminus x, R) \simeq R$

2) R-orientation of M is compatible choice of R-orientations at all $x \in M$.

Meaning of compatible



$$H_n(M, M \setminus x; R) \xleftarrow{\simeq} H_n(M, M \setminus B; R) \xrightarrow{\simeq} H_n(M, M - y; R)$$

$$\mu_x \longleftrightarrow \mu_y$$

Bundle of orientations   classical case: $R = \mathbb{Z}$

$$\tilde{M} = \{ (x, \mu_x) \mid x \in M \quad \mu_x \text{ gen of } H_n(M, M \setminus x) \}$$

$\pi \downarrow$ ← two fold cover $= \mathbb{Z}/2$-principal bundle

M

Exer Topologize $\tilde{M}$ so that π is indeed covering
"nearby (x, μ_x) are compatible"

Ex $M = \mathbb{RP}^2 \rightsquigarrow \tilde{M} = S^2$

$M = S^2 \rightsquigarrow \tilde{M} = S^2 \amalg S^2$

Observe: $\tilde{M}$ has a tautological orientation

$(x, \mu_x) \rightsquigarrow \mu_x$

Prop Orientations of M $\longleftrightarrow$ sections s of $\tilde{M} \xrightarrow{\pi} M$

Pf. Exercise.

Cor M simply-conn ($\text{conn}, \pi_1 = \text{triv}$) $\Rightarrow$ M is orientable with exactly <u>two orientations</u>.

Rmk: There's an analogous covering space for R-orientation

Def. Fund <u>class</u> $[M] \in H_n(M, R)$ such that for all $x \in M$

$$H_n(M, R) \longrightarrow H_n(M, M - x, R)$$

we have $[M] \longmapsto$ generator (local orientation)

In other words $[M]$ defines an R-orientation of M.

Conversely...

Thm M compact conn n-mfd

1) M R-orientable $\Rightarrow$ $H_n(M, R) \xrightarrow{\sim} H_n(M, M - x, R) \cong R$ for all $x \in M$
2) M not R-orientable $\Rightarrow$ $H_n(M, R) \underset{\text{inj.}}{\hookrightarrow} H_n(M, M - x, R) \cong R$ for all $x \in M$ with image $\{r \in R \mid 2r = 0\}$
3) $H_i(M, R) = 0 \quad i > n$

Cor $\underline{R=\mathbb{Z}}$ $H_n(M,\mathbb{Z}) = \begin{cases} \mathbb{Z} & M \text{ orientable} \Leftarrow 1) \\ 0 & M \text{ not orientable} \Leftarrow 2) \end{cases}$

$\underline{R=\mathbb{Z}/2}$ $H_n(M,\mathbb{Z}/2) = \mathbb{Z}/2 \Leftarrow 1)$

To prove Thm, we'll use:

Let $M_R = \{(x, r_x) \mid x \in M,\ r_x \in H_n(M, M\setminus x, R)\}$, $\pi: M_R \to M$

(r_x: not nec gen.)

Covering space with fibers $\cong R$ (in fact a covering space$/M$)

Sections = compatible choices of elts in local homol.

Lemma M n-mfd, $A \subset M$ compact

a) α a section of $\pi: M_R \to M$ $\Rightarrow \exists!\ \alpha_A \in H_n(M, M\setminus A)$ restricting to α_x for all $x \in A$ under $H_n(M, M\setminus A) \to H_n(M, M\setminus x)$

b) $H_i(M, M\setminus A; R) = 0$ $i > n$

Lemma $\Rightarrow$ Thm M compact, set $A = M$

So b) $\Rightarrow$ 3)

For 1) and 2), consider natural map

$$H_n(M,R) \to \text{Sections}\left(\pi: M_R \to M\right)$$

a) $\Rightarrow$ isom.

Since M conn, 1), 2) follow since sections determined by restriction to a point.

We'll leave proof of Lemma to Office Hours or other discussions ...

<u>Thm</u> (Poincaré duality)

M compact, R-orientable n-mfd

Let $[M] \in H_n(M,R)$ be a fund class

Then
$$H^k(M,R) \xrightarrow{\sim} H_{n-k}(M,R)$$
$$\varphi \longmapsto [M] \frown \varphi$$