215a Lecture 22 (M 11/16/20) Cohomology

<u>Analogy</u> $H_*(X)$ $\qquad\qquad$ $H^*(X)$

"loc const. distributions" $\qquad$ "loc const functions"

<u>Def.</u> (C_*, ∂) chain complex of free ab gps

G ab gp (coefficients)

$\rightsquigarrow$ $(C^*(G), \delta)$ <u>dual</u> cochain complex

$$\mathrm{Hom}(C_0, G) \xrightarrow{\delta_0} \mathrm{Hom}(C_1, G) \xrightarrow{\delta_1} \mathrm{Hom}(C_2, G) \xrightarrow{\delta_2} \cdots$$

$$(\delta_n \varphi)(c) := \varphi(\partial c)$$

<u>Note</u> $\delta^2 = 0$ b/c $\partial^2 = 0$

$\rightsquigarrow$ $H^n(C_\cdot, G) = \ker(\delta_n) / \operatorname{im}(\delta_{n-1})$

<u>Rmk</u> Functorial $\alpha : C_\cdot \to D_\cdot$ $\rightsquigarrow$ $\alpha^* : H^*(D_\cdot, G) \to H^*(C_\cdot, G)$

<u>Application</u>: X top sp, $H^*(X, G) = H^*(C_*(X); G)$

sing cohom. $\qquad$ sing chains

Likewise simplicial, cellular cohomology.

<u>Ex</u> (funct. analysis) $X = \mathbb{Z}$ $\quad \cdots \bullet \; \bullet \; \bullet \; \bullet \cdots$

$0 - \cdots 0 \; -3 \; 0 \; 2 \; 0 \cdots - 0$

$$C_*^{CW}(X) = \bigoplus_{n \in \mathbb{Z}} \mathbb{Z} \cdot p_n$$

elts have only fin many $\neq 0$ terms

$$C^*_{cw}(X) = \prod_{n \in \mathbb{Z}} \mathbb{Z} \cdot \tilde{p}_n$$

$\cdots 1 \; -1 \; 1 \; -1 \cdots$

↖ elts are arb. seqs.

$\underline{\text{Ex}}$ $X = \mathbb{RP}^n$ cellular cohom

$$\begin{array}{lcccccc} & 0 & 1 & 2 & 3 & \cdots & n \\ C_*(X) & \mathbb{Z} \xleftarrow{0} & \mathbb{Z} \xleftarrow{2} & \mathbb{Z} \xleftarrow{0} & \mathbb{Z} & \cdots \leftarrow & \mathbb{Z} \\ H_*(\mathbb{RP}^n) & \mathbb{Z} & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & \cdots & \end{array}$$

$$C^*(X, \mathbb{Z}) \qquad \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \cdots \to \mathbb{Z}$$

$$\begin{array}{llcccccccc} & & 0 & 1 & 2 & 3 & & & & \\ \underline{n \text{ odd}} & H^*(\mathbb{RP}^n) & \mathbb{Z} & 0 & \mathbb{Z}/2 & 0 & \cdots & \mathbb{Z}/2 & \mathbb{Z} & \\ \underline{n \text{ even}} & & \mathbb{Z} & 0 & \mathbb{Z}/2 & 0 & \cdots & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 \end{array}$$

$$C^*(X; \mathbb{Z}/2) \qquad \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{?} \cdots \to \mathbb{Z}/2$$

$$H^*(\mathbb{RP}^n; \mathbb{Z}/2) \qquad \mathbb{Z}/2 \qquad \mathbb{Z}/2 \qquad \cdots \qquad \mathbb{Z}/2$$

$/2 \triangle$ $$H^*(C_\cdot, G) \neq \mathrm{Hom}(H_*(C_\cdot), G)$$

$\underline{\text{Thm}}$ (Universal Coeff. Thm) - algebraic (not topological)

Splittable natural SES in chain complexes

$$0 \to \mathrm{Ext}^1(H_{n-1}(C_\cdot), G) \to H^n(C_\cdot, G) \xrightarrow{h} \mathrm{Hom}(H_n(C_\cdot), G) \to 0$$

$\cong$ $\oplus$ (but not naturally)

Cor $\alpha : C_\cdot \to D_\cdot$ induces an isom on H_*

$\Rightarrow$ $\alpha^* : C^*(D_\cdot, G) \to C^*(C_\cdot, G)$ induces an isom on H^*.

Rmk What is h? $[\varphi] \in H^n(C_\cdot, G)$ is rep by a map

$$\varphi : C_n \longrightarrow G \quad \text{such that } (\delta\varphi)(c) = 0$$

$$\text{i.e. } \varphi(\partial c) = 0$$

In other words φ descends $\overline{\varphi} : C_n/B_n \longrightarrow G$

$$\begin{array}{c} C_n/B_n \\ \cup \\ Z_n/B_n \\ \parallel \\ H_n \end{array}$$

$$h(\varphi) = \overline{\varphi}\big|_{Z_n/B_n = H_n}$$

Rmk: There's also a univ coeff thm for $H_n(C_* \otimes G)$ in terms of $H_n(C_*) \otimes G$ and $\mathrm{Tor}_1(H_{n-1}(C_*), G)$

$$0 \to H_n(C_*) \otimes G \to H_n(C_* \otimes G) \to \mathrm{Tor}_1(H_{n-1}(C_*), G) \to 0$$

What is Ext^1 (for ab gps)?

$$0 \to A \to B \to C \to 0 \quad \text{SES}$$

$$\mathrm{Hom}(A,G) \leftarrow \mathrm{Hom}(B,G) \leftarrow \mathrm{Hom}(C,G) \leftarrow 0 \quad \text{exact seq.}$$

not surj in general!

Ext^1 will capture the interesting cokernel

Suppose $F_\bullet \to C$ is free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to C \to 0 \quad \text{exact seq.}$$

with all F_i free ab gps.

In particular, we always have

$$0 \to \mathrm{Ker} \to F(C) \to C \to 0$$

Note Ker free also; $F(C)$ free ab gp on C

Def. Apply $\mathrm{Hom}(-, G)$ to any free resolution $F_\bullet \to C$

$$\mathrm{Ext}^n(C, G) := H^n(\mathrm{Hom}(F_\bullet, G))$$

Exer Ext^n is naturally indep. of choice of resolution.

Consequence $\mathrm{Ext}^{n>1} \simeq 0$ since we have resolution

$$0 \to \mathrm{Ker} \to F(C) \to C \to 0$$

Exer $\operatorname{Ext}^0 \cong \operatorname{Hom}$.

Note

$$0 \leftarrow \underset{\displaystyle H^1}{\overset{\displaystyle H^1}{}} \operatorname{Ext}^1(C,G) \leftarrow \left[\operatorname{Hom}(\operatorname{Ker}, G) \leftarrow \operatorname{Hom}(F(C), G)\right] \leftarrow \operatorname{Hom}(C,G) \leftarrow 0$$

$$H^1 = \operatorname{Ext}^1(C,G), \qquad \operatorname{Hom}(C,G) = \operatorname{Ext}^0(C,G) = H^0$$

Exer $\operatorname{Ext}^1(C,G) =$ Isom classes of extensions (SESs)

$$0 \to G \to \boxed{} \to C \to 0$$

Basic calculations

1) Additivity $\operatorname{Ext}^1(C \oplus C', G) \cong \operatorname{Ext}^1(C,G) \oplus \operatorname{Ext}^1(C',G)$

2) Vanishes on free $\operatorname{Ext}^1(\underset{\text{free}}{C}, G) \cong 0$

3) Torsion behavior $\operatorname{Ext}^1(\mathbb{Z}/m, G) \cong G/mG$

$$0 \to \left[\mathbb{Z} \xrightarrow{m} \mathbb{Z}\right] \to \mathbb{Z}/m \to 0$$

$$\operatorname{Ext}^1(\mathbb{Z}/m, G) \leftarrow \left[G \xleftarrow{m} G\right] \leftarrow \operatorname{Hom}(\mathbb{Z}/m, G)$$

$\Rightarrow$ C fin gen then $\operatorname{Ext}^1(C, \mathbb{Z}) = \operatorname{Tors}(C)$

Back to U.C.T. - let's see how it works for $\mathbb{RP}^5$

$$
\begin{array}{lcccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
C_\cdot & \mathbb{Z} \xleftarrow{0} & \mathbb{Z} \xleftarrow{2} & \mathbb{Z} \xleftarrow{0} & \mathbb{Z} \xleftarrow{2} & \mathbb{Z} \xleftarrow{0} & \mathbb{Z} \\
H_* & \mathbb{Z} & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z} \\
H^* \simeq \mathrm{Hom}(H_*, \mathbb{Z}) & \mathbb{Z} & \langle 0 \rangle & \langle 0 \rangle & \langle 0 \rangle & \langle 0 \rangle & \mathbb{Z} \\
\oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus \\
\mathrm{Ext}^1(H_{*-1}, \mathbb{Z}) & \langle 0 \rangle & \langle 0 \rangle & \mathbb{Z}/2 & \langle 0 \rangle & \mathbb{Z}/2 & \langle 0 \rangle \\
 & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel \\
 & \mathbb{Z} & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & \mathbb{Z}
\end{array}
$$

<u>Cor (of UCT)</u> $H_*(C.)$ fin gen $\Rightarrow H^n(C_\cdot, \mathbb{Z}) \simeq H_n(C.)/\mathrm{Tors}_n \oplus \mathrm{Tors}_{n-1}$

Begin proof of UCT:

Lemma There's ism of SES's

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ker}(h) & \to & H^n(C_\cdot, G) & \xrightarrow{h} & \mathrm{Hom}(H_n(C_\cdot), G) & \to & 0 \\ & & \cong & & \| & & \cong & & \\ 0 & \to & \mathrm{Coker}(i_{n-1}^*) & \to & H^n(C_\cdot, G) & \to & \mathrm{Ker}(i_n^*) & \to & 0 \end{array}$$

where $i_n : B_n \hookrightarrow Z_n$ n-bdies into n-cycles in C_n

Pf. Exer: Show h we constructed is surj.
(Use $C_\cdot$ is chain complex of free ab gps)

We'll start next time constructing bottom SES and ism.