215a Lecture 19 (M 11/2/20) Eilenberg - Steenrod axioms + Hurewicz Theorem

E-S axioms for a "homology theory"

Functor $h_* : \mathrm{TopPairs} \longrightarrow \mathbb{Z}\text{-gr ab. gps}$
$(X,A) \longmapsto h_*(X,A)$

Note $\mathrm{Top} \subset \mathrm{TopPairs}$
$A \mapsto (A, \emptyset)$
$X \mapsto (X, \emptyset)$

$+$ nat transf $\partial : h_*(X,A) \longrightarrow h_{*-1}(A)$

providing LES: $\partial \circlearrowright h_*(A) \xrightarrow{i_*} h_*(X) \xrightarrow{j_*} h_*(X,A) \circlearrowright \partial$

provided by functoriality

satisfying:

1) Homotopy invariance $f \sim g \Rightarrow f_* = g_*$

Note $f_* = h_*(f)$
$g_* = h_*(g)$

2) Excision $Z \subset A$ with $\bar{Z} \subset \mathrm{Int}(A)$

$$\Rightarrow h_*(X \setminus Z, A \setminus Z) \xrightarrow{\sim} h_*(X,A)$$

nat. map.

3) Additivity $X = \coprod_\alpha X_\alpha \Rightarrow h_*(X) \xleftarrow{\sim} \bigoplus_\alpha h_*(X_\alpha)$

nat map

(Rmk: for finite disj unions, this already follows.)

4) Dimension $h_*(pt) = \begin{cases} C \text{ ab gp} & * = 0 \\ 0 & \text{else} \end{cases}$

Basic choice
$C = \mathbb{Z}$.

Generalized homol. theories

Allow non discrete ab gps as homol. of pt

Reduced version $\tilde{h}_* : \text{Nonempty Top} \to \mathbb{Z}\text{-gr ab. gps}$

$$\tilde{h}_*(X) = \ker\left(h_*(X) \xrightarrow{\text{deg.}} h_*(pt) \right) \qquad X \text{ nonempty sp.}$$

$$h_*(Y) = \tilde{h}_*(Y \amalg pt) \qquad Y \text{ space}$$

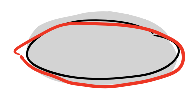
3') $X = \bigvee_\alpha X_\alpha \Rightarrow \tilde{h}_*(X) \xleftarrow{\sim} \bigoplus_\alpha \tilde{h}_*(X_\alpha)$

4') $\tilde{h}_*(S^0) = \begin{cases} \mathbb{C} & * = 0 \\ 0 & \text{else} \end{cases}$

Ex Let's show $h_*(S^n) = \begin{cases} \mathbb{C} & * = 0, n \\ 0 & \text{else.} \end{cases}$

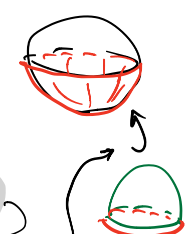
$n > 0$

Consider LES of $(D^n, \partial D^n = S^{n-1})$:



$$\begin{array}{ll} n & \underset{0}{h_n(D^n)} \to h_n(D^n, S^{n-1}) \\ & \qquad \cong \\ n-1 & \to \underset{= \mathbb{C}}{h_{n-1}(S^{n-1})} \to \underset{= 0}{h_{n-1}(D^n)} \to \end{array}$$

Consider also LES of (S^n, D^n)



$$\begin{array}{ll} n & \underset{0}{h_n(D^n)} \to h_n(S^n) \xrightarrow{\sim} h_n(S^n, D^n) \\ n-1 & \to \underset{= 0}{h_{n-1}(D^n)} \end{array}$$

$$h_n(S^n, D^n) \overset{\text{excision}}{\cong} h_n(D^n, \partial D^n)$$

Thm For CW pairs $h_*(X,A) \cong H_*((X,A); h_0(pt))$

functorially + nat bdy map

Idea of Proof By LES suffices to consider $X = (X, \emptyset)$

Can construct cellular version of h_*:

$$\cdots \dashrightarrow h_*(X^{n+1}, X^n) \to h_*(X^n, X^{n-1}) \to h_*(X^{n-1}, X^{n-2}) \to \cdots$$

and show it calculates 1) $h_*(X)$

2) chain groups are $\bigoplus_{n\text{-cells}} h_0(pt)$

What remains to show is bdy map is as expected:

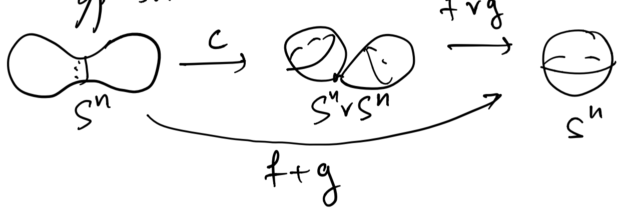
Need: $f: S^n \to S^n$ of deg $d \Rightarrow h_*(f) =$ mult by d on $h_0(pt)$

Note: f deg $= \begin{cases} 0 \\ 1 \end{cases} \Rightarrow h_*(f) = \begin{cases} 0 \\ id. \end{cases}$

So need to check for other degs in $\mathbb{Z}$.

Use fact: $\pi_n(S^n) \cong \mathbb{Z}$ homot. classes of based maps. $S^n \to S^n$

gp structure, gps



So suffices to check h_* additive wrt. group str.

Lemma $\tilde{h}_*(f+g) = \tilde{h}_*(f) + \tilde{h}_*(g)$

Proof

$$x \longmapsto \tilde{h}_*(f+g)(x)$$

$$\begin{tikzcd}
\tilde{h}_*(S^n) \arrow[r, "c_*"] & \tilde{h}_*(S^n \vee S^n) \arrow[r, "(f \vee g)_*"] & \tilde{h}_*(S^n) \\
& \tilde{h}_*(S^n) \oplus \tilde{h}_*(S^n) \arrow[u, "\cong"'] &
\end{tikzcd}$$

$$x \longmapsto (x, x) \qquad (x, 0) \longmapsto x \qquad (0, x) \longmapsto x$$

$$\tilde{h}_*(f+g)(x) = \big((f \vee g)_* \circ c_*\big)(x) = \tilde{h}_*(f)(x) + \tilde{h}_*(g)(x)$$

∎

Conclusion $h_*(X) \cong H_*(X, h_0(pt))$ for CW X.

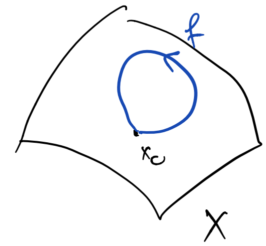
On maps? take $f: X \to Y$ and apply cellular approx and so obtain induced map on cellular hom.

. . . ∎

Hurewicz Theorem ($n=1$)

Based space (X, x_0) path-conn

Natural homos
$$\begin{tikzcd} \pi_1(X,x_0) \arrow[rr, "h"] \arrow[dr] & & H_1(X) \\ & \pi_1^{ab}(X,x_0) = \pi_1/[\pi_1,\pi_1] \arrow[ur, "\cong"'] & \end{tikzcd}$$



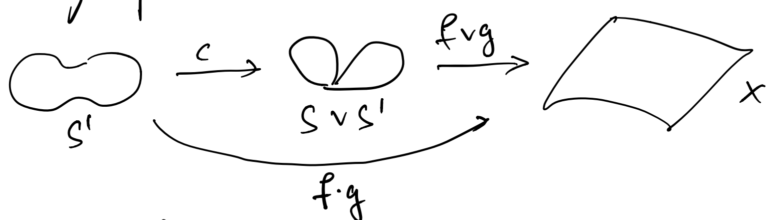
<u>Proof</u> $f : (I, \partial I) \to (X, x_0)$ regard as cycle $h(f)$

More scientifically $f : S^1 \to X \rightsquigarrow f_* : H_1(S^1) \to H_1(X)$

$$\mathbb{Z} \ni 1 \longmapsto h(f)$$

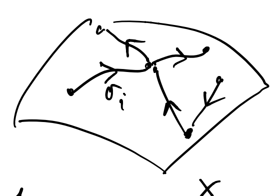
Note $f \simeq g \Rightarrow f_* = g_*$ so $h(f) = h(g)$

<u>Exer</u> Check group homo – same arg as Lemma above!



Show $h(f \cdot g) = h(f) + h(g)$

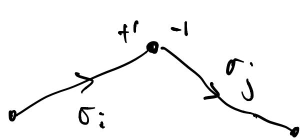
<u>Need</u>: $\pi_1^{ab} \to H_1$ is isom.



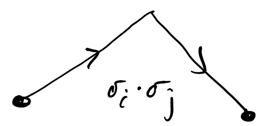
<u>Surj</u>: $z = \sum_i n_i \sigma_i$ 1-cycle in X

Relabel $z = \sum_i \sigma_i$ ie so each $n_i = 1$.

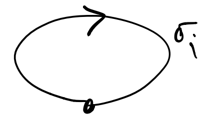
$\partial(z) = 0 \Rightarrow$ for σ_i must have σ_j as pictured:



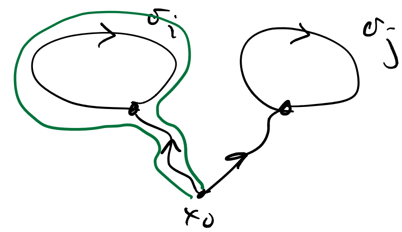
in this case can combine



Can do this until each σ_i is closed loop.



Finally connect each to basept
(X path conn)

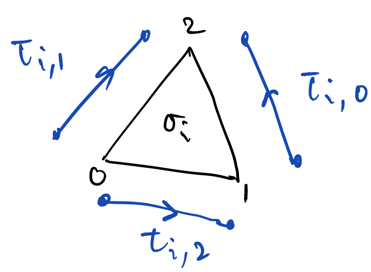


Now $\sum \sigma_i$ is in image under h of a loop.

<u>Inj</u> <u>Need</u>: $f \in \mathrm{Ker}(h) \Rightarrow f = 0$ in π_1^{ab}

$$h(f) = \partial\left(\sum_i n_i \sigma_i\right)$$

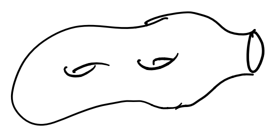
Again can relabel so that say all $n_i = \pm 1$



$$h(f) = \partial\left(\sum_i n_i \sigma_i\right) = \sum_{i,j} n_i (-1)^j \tau_{i,j}$$

Pair off cancelling $T_{i,j}$ to obtain a 2-dim simpl. complex K with a map $\sigma : K \to X$ with bdy $f : \partial K \to X$

Exer K is orientable surface with single bdy circle!



we know class of this loop is a commutator!

Conclude: $f = 0$ in π_1^{ab} !

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