215A Lecture 13 (M 10/12/20): Singular homology

Recall Δ-complex X

1) $\sigma_\alpha : \Delta^n \longrightarrow X$

$\sigma_\alpha|_{\mathring{\Delta}^n}$ injections, partition X

2) $\sigma_\alpha|_{\partial_i \Delta^n} = \sigma_\beta$ some β

3) $A \subset X$ open $\Leftrightarrow$ $\sigma_\alpha^{-1}(A)$ open all α.

Recall as well: Δ-homology chain complex

$$\Delta_0(X) \xleftarrow{\partial_1} \Delta_1(X) \xleftarrow{\partial_2} \Delta_2(X) \quad \cdots$$

$\Delta_n(X) = \{$free ab. group on $\sigma_\alpha : \Delta^n \to X\}$ n-chains

$$H_n^\Delta(X) = \underbrace{\ker(\partial_n)}_{n\text{-cycles}} / \underbrace{\operatorname{im}(\partial_{n+1})}_{n\text{-boundaries}}$$ n-homology classes

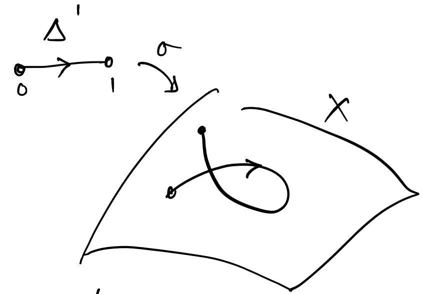
Issue: $H_*^\Delta(X)$ not evidently indep of Δ-str.

Also: not evidently functorial except for very special maps comp. with Δ-strs.

Correction: Singular homology!

Price: Much more complicated to calculate directly

<u>Idea</u>: Allow <u>all</u> simplices!



<u>Def</u>. X top sp.

1) <u>Sing</u>. <u>n-simplex</u> $\sigma : \Delta^n \to X$

2) <u>Sing</u> <u>n-chain</u>: fin formal lin comb of sing n-simplices

$C_n(X)$ $\qquad \sum_\alpha n_\alpha \sigma_\alpha, \quad n_\alpha \in \mathbb{Z}$.

3) <u>Sing</u> <u>bdy</u> <u>op</u>: $\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{\partial_i \Delta^n}$

extend to be homomorphism

Same arg as before: $\partial_n \circ \partial_{n+1} = 0$.

4) <u>Sing</u> <u>chain complex</u>

$$C_0(X) \xleftarrow{\partial_1} C_1(X) \xleftarrow{\partial_2} C_2(X) \cdots$$

5) <u>Sing</u>. <u>homology</u>

$$H_n(X) = \ker(\partial_n) / \operatorname{im}(\partial_{n+1})$$

<u>Immediate observations</u> • $H_n(X)$ <u>homeo</u> <u>invt</u>!

(We will see: $H_n(X)$ homot. invt!)

• <u>Evident</u> <u>functoriality</u> $f : X \to Y$

$$\begin{array}{ccccc} C_0(X) & \xleftarrow{\partial_1} & C_1(X) & \xleftarrow{\partial_2} & \cdots \\ \downarrow & \circ & \downarrow & \circ & \cdots \\ C_0(Y) & \xleftarrow{\partial_1} & C_1(Y) & \xleftarrow{\partial_2} & \cdots \end{array}$$

$f_\#$ map of chain complexes

$f_\#(\sigma) = \sigma \circ f$ extend to be homomorphism

Induced map on homology $f_* : H_n(X) \to H_n(Y)$ all n.

<u>Lemma</u> $H_n(pt) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & \text{else} \end{cases}$

<u>Pf</u>. Sing chain complex: since unique map $\Delta^n \to pt$ all n

$$\overset{0}{\mathbb{Z}} \xleftarrow{0_*} \overset{1}{\mathbb{Z}} \xleftarrow[1\cdot]{\sim} \overset{2}{\mathbb{Z}} \xleftarrow{0\cdot} \overset{3}{\mathbb{Z}} \xleftarrow{\sim} \cdots$$

$$0_* = \sum_{i=0}^{1}(-1)^i \qquad 1\cdot = \sum_{i=0}^{2}(-1)^i \qquad 0\cdot = \sum_{i=0}^{3}(-1)^i \quad \cdots$$

$\boxtimes$

<u>Prop</u> 1) $X = \coprod_\alpha X_\alpha$ disj union of path-comps

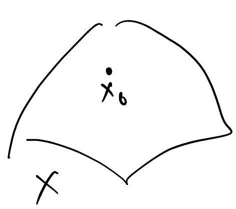
Then $H_n(X) = \bigoplus_\alpha H_n(X_\alpha)$

2) X path-conn, <u>nonempty</u>

Then $H_0(X) \xrightarrow{\sim} \mathbb{Z}$.

$$\sum_\alpha n_\alpha \sigma_\alpha \mapsto \sum n_\alpha$$

<u>Pf</u>. 1) Follows from fact that Δ^n is path-conn so maps $\sigma : \Delta^n \to X$ lie in single path-comp.

2) 

Denote by $\varepsilon : C_0(X) \to \mathbb{Z}$ "degree map"

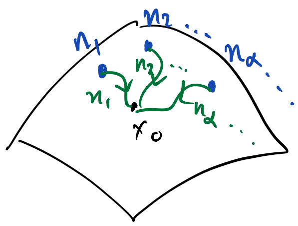
$$\sum_\alpha n_\alpha \sigma_\alpha \mapsto \sum_\alpha n_\alpha$$

Note ε descends to $\varepsilon : H_0(X) \to \mathbb{Z}$

since $\varepsilon(\partial\sigma) = \varepsilon(+1\cdot\sigma|_{\partial_0\Delta^1} + (-1)\cdot\sigma|_{\partial_1\Delta^1})$ (σ ↑ 1-simplex)

$= 1 + (-1) = 0$.

Note ε surjective $\varepsilon(1\cdot x_0) = 1$.
(since X nonempty!)



Observe any 0-cycle $\sum_\alpha n_\alpha \sigma_\alpha \sim \left(\sum_\alpha n_\alpha\right)\cdot \sigma_0$ (homologous)

where $\sigma_0 : \Delta^0 \longrightarrow \{x_0\} \in X$ ∎

So $\varepsilon : H_0(X) \to \mathbb{Z}$ is injective

$n\cdot\sigma_0 \longmapsto n$

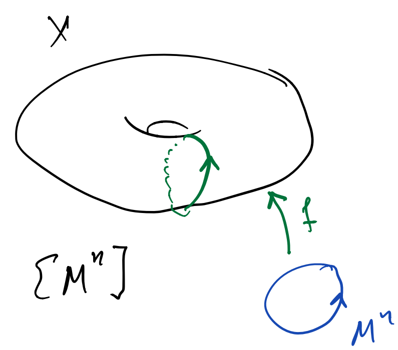
Terminology Reduced homology of nonempty X

$$\mathbb{Z} \overset{\varepsilon}{\twoheadleftarrow} C_0(X) \leftarrow C_1(X) \leftarrow C_2(X) \leftarrow \cdots$$

Define $\tilde{H}_n(X)$ to be homol. of this complex.

Ex. $\tilde{H}_n(pt) = 0$ all n.

Informal remark: constructing some cycles.



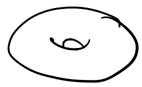
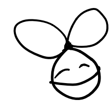
Observe: suppose M^n is a conn, compact, oriented n-manifold

Then $H_n(M^n) = \mathbb{Z}$ gen by $[M^n]$
(same n)

Given a map $f : M^n \longrightarrow X$, by functoriality we obtain $f_* : H_n(M^n) \to H_n(X)$

and so a class $f_*[M^n] \in H_n(X)$

Caution Not all classes arise in this way.

Ex T^2  vs $S^1 \vee S^1 \vee S^2$ 

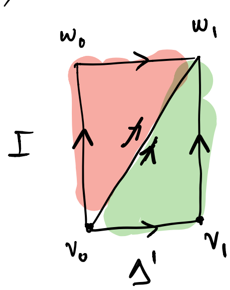
T^2	$S^1 \vee S^1 \vee S^2$
$\pi_1 = \mathbb{Z}^2$	$\pi_1 = F^2$
$H_0 = \mathbb{Z}$	$H_0 = \mathbb{Z}$
$H_1 = \mathbb{Z}^2$	$H_1 = \mathbb{Z}^2$
$H_2 = \mathbb{Z}$	$H_2 = \mathbb{Z}$

Thm $f \simeq g : X \to Y$ (homotopic) $\Rightarrow$ $f_* = g_* : H_n(X) \to H_n(Y)$ all n.

Cor $f : X \to Y$ homot equiv $\Rightarrow$ $f_* : H_n(X) \xrightarrow{\sim} H_n(Y)$ all n.

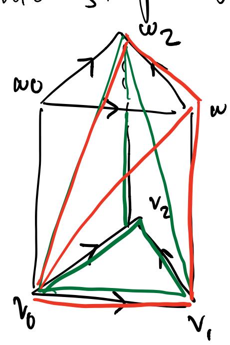
To prove Thm: some ingredients, motivated by $F : X \times I \to Y$, $F_0 = f$, $F_1 = g$

1) $\Delta^n \times I$ "prism" cut up into simplices



$[v_0, v_1 \mid w_1]$

$[v_0 \mid w_0, w_1]$



$[v_0, v_1, v_2 \mid w_2]$

$[v_0, v_1 \mid w_1, w_2]$

$[v_0 \mid w_0, w_1, w_2]$

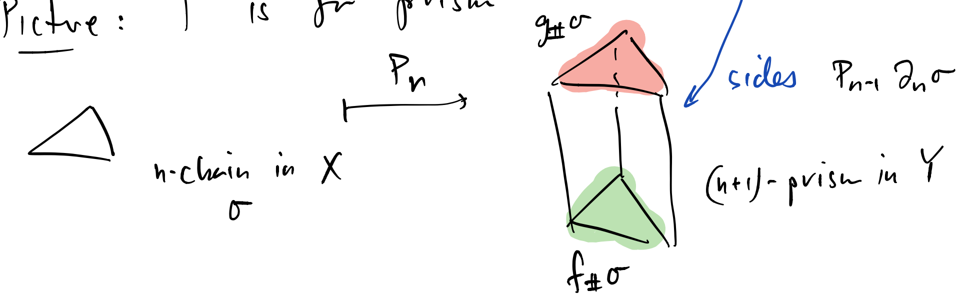
$\Delta^n \times I$ cut up into $n+1$ $(n+1)$-simplices $[v_0, \dots, v_i \mid w_i, \dots, w_n]$

2) Given $F : X \times I \to Y$, construct chain homotopy

$$P_n : C_n(X) \to C_{n+1}(Y)$$

Key property $\quad \partial_{n+1} P_n = g_\# - f_\# - P_{n-1} \partial_n$

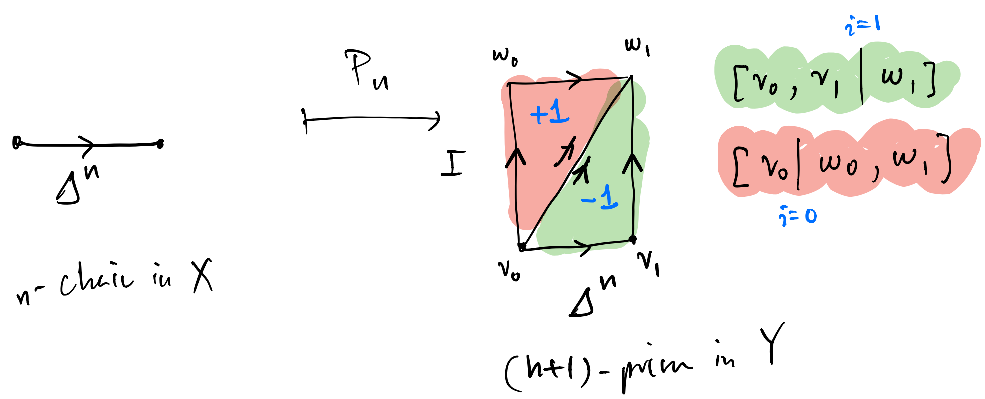
Picture: "P is for prism"



Definition of P_n:

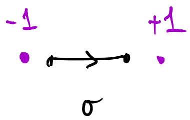
(n+1)-simplices from 1)

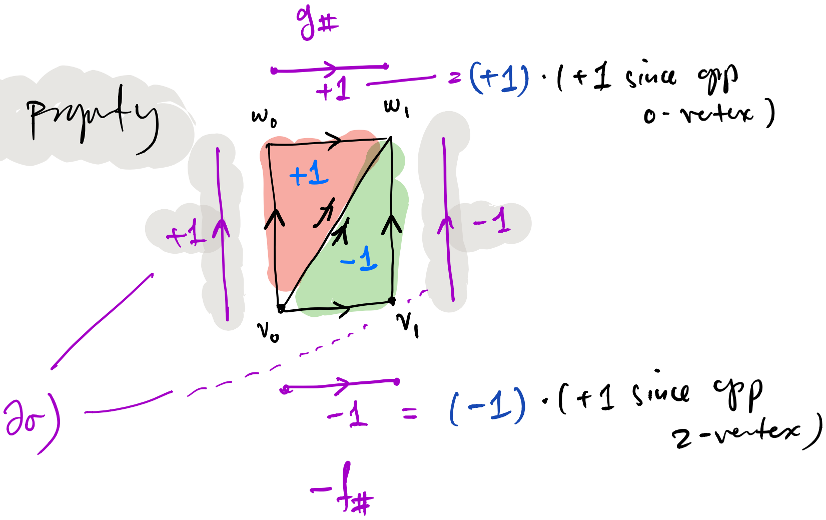
$$P_n(\sigma) = \sum_{i=0}^{n} (-1)^i \, F \circ (\sigma \times id)\Big|_{[v_0, \dots, v_i | w_i, \dots, w_n]}$$



Exer verify key property

n = 1 :





3) Chain homotopy P between $f_\#$, $g_\#$ induces $f_* = g_*$ on homology.

$$
\begin{array}{ccccccc}
C_0(X) & \leftarrow & C_1(X) & \leftarrow & C_2(X) & \leftarrow & \cdots \\
\downarrow & \searrow^{P_0} & \downarrow & \searrow^{P_1} & \downarrow & \searrow^{P_2} & \\
C_0(Y) & \leftarrow & C_1(Y) & \leftarrow & C_2(Y) & \leftarrow & \cdots
\end{array}
\qquad f_\#, g_\#
$$

Purely alg fact about maps of chain complexes.

Pf of Thm 1) + 2) + 3) Done!