215A Lecture 12 (W 10/7/20) Homology! (first simplicial)

X top sp $\rightsquigarrow H_*(X)$ $\mathbb{Z}_{\geq 0}$-graded abelian group

$* = 0, 1, 2, \ldots$

(functorial)

We'll see: homotopy invariant.

Rough Idea: "linearize" space X

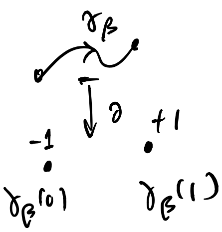
$X \rightsquigarrow$ Formal finite $\mathbb{Z}$-lin combs of pts of X

$$\sum_\alpha n_\alpha x_\alpha$$

"Distributions of homol. deg 0"

Two such lin combs are equiv if they differ by ∂ of formal finite $\mathbb{Z}$-lin comb of paths in X

$$\sum_\beta n_\beta \gamma_\beta$$

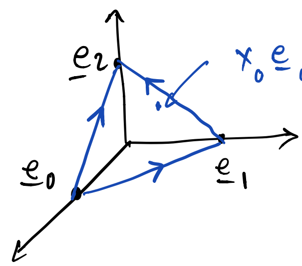


"Dists of homol deg 1"

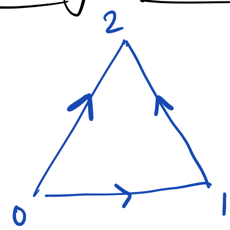
$\vdots$

<u>Def</u> <u>n-simplex</u> $\Delta^n = \mathrm{Conv}\{\underline{e}_0, \dots, \underline{e}_n\} \subset \mathbb{R}^{n+1}$

$$= \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1}_{\geq 0} \mid \sum x_i = 1\}$$



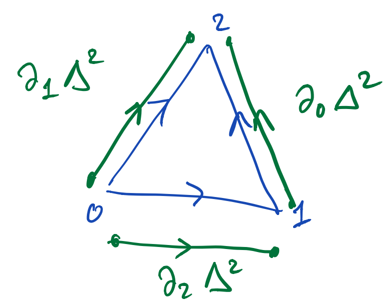
Include <u>ordering</u> <u>on vertices</u> as part of structure



<u>Notation</u> $\mathring{\Delta}^n$ = interior , $\partial\Delta^n$ = boundary

$$\partial\Delta^n = \bigcup_{i=0}^{n} \partial_i \Delta^n$$ ← boundary face opp. ith vertex

<u>Note</u> $\partial_i\Delta^n$ inherits ordering of its vertices



Def <u>Δ-complex</u> X (special case of CW complex)

is a topol sp X with

1) $\sigma_\alpha : \Delta^n \longrightarrow X$ with $\sigma_\alpha|_{\mathring{\Delta}^n} : \mathring{\Delta}^n \underset{\text{inj.}}{\hookrightarrow} X$

   and $\sigma_\alpha|_{\mathring{\Delta}^n}$ images partition X

2) $\sigma_\alpha|_{\partial_i \Delta^n} = \sigma_\beta$ for some β

3) $A \subset X$ open $\Longleftrightarrow$ $\sigma_\alpha^{-1}(A)$ open all α

Def <u>Simplicial complex</u> (special case of Δ-complex)

1) $\sigma_\alpha : \Delta^n \underset{\text{inj}}{\hookrightarrow} X$ with $\sigma_\alpha|_{\mathring{\Delta}^n}$ images partition X

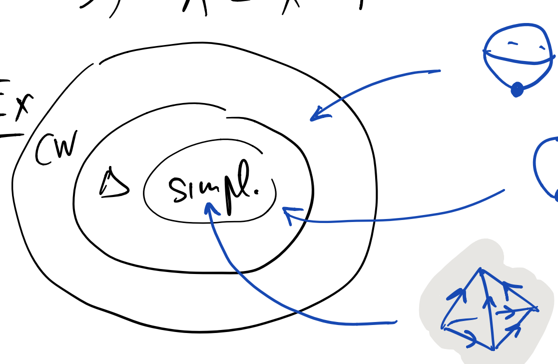
2) $\sigma_\alpha(\Delta^n) \cap \sigma_\beta(\Delta^m) = \sigma_\gamma(\Delta^\ell)$ some γ

3) $A \subset X$ open $\Longleftrightarrow$ $\sigma_\alpha^{-1}(A)$ open all α

Simpl. Complex

Determined by vertices + ordered subsets

<u>Ex</u>



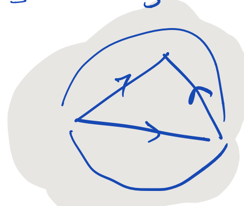
$S^2 = e_0 \cup e_2$

$S^1 = \Delta^0 \cup \Delta^1$

$S^2 =$ union of 4 Δ^2's.

<u>Exer</u> Find smallest CW, Δ, simpl. presentations of $\mathbb{RP}^2, T^2, \ldots$ surfaces.

Δ-complex S^2 : $S^2 = \Delta^2 \underset{\partial \Delta^2}{\amalg} \Delta^2$



Def. X Δ-complex

1) $\Delta_n(X) = \{$ fin lin combs $\sum_\alpha n_\alpha \sigma_\alpha$ of n-simplices of X $\}$

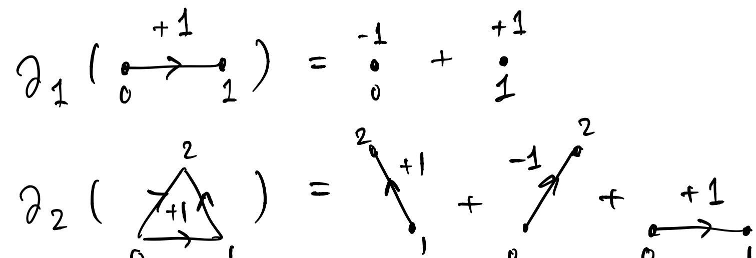
   (nth <u>chain group</u>
   free abelian with basis σ_α)

2) $\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$ homo. defined by

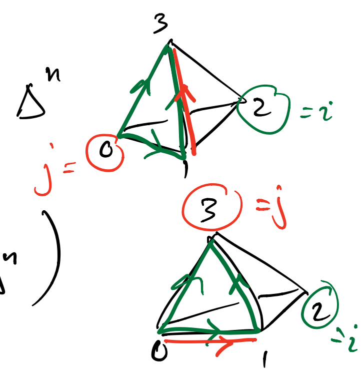
   (nth <u>boundary</u> map)

$$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \, \sigma_\alpha\big|_{\partial_i \Delta^n}$$

Ex



$$\boxed{\textbf{Lemma} \quad \partial_{n-1} \circ \partial_n = 0 \ !}$$



Proof Check on basis elts.

$$\partial_{n-1}(\partial_n(\sigma_\alpha)) = \partial_{n-1}\Big(\sum_i (-1)^i \sigma_\alpha\big|_{\partial_i\Delta^n} \Big)$$

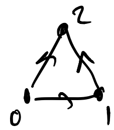
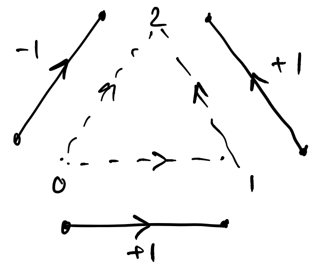
$$= \sum_{j<i} (-1)^i (-1)^j \, \sigma_\alpha\big|_{\partial_j \partial_i \Delta^n}$$

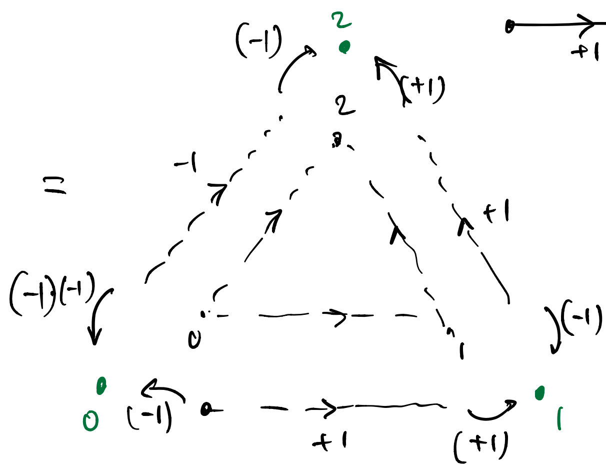
$$+ \sum_{j>i} (-1)^i (-1)^{j-1} \, \sigma_\alpha\big|_{\partial_j \partial_i \Delta^n}$$

$$= 0 \qquad \blacksquare$$

cancel each other!
reindex $i \leftrightarrow j$

($\partial_j \partial_i \Delta^n$ = cod 2 face opp edge i—j)

$\underline{\text{Ex}}$ $\partial_1\big(\partial_2\big($  $\big)\big) = \partial_1\Big($  $\Big)$

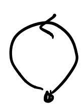
$=$ 

$\underline{\text{Def}}$ $\underline{\Delta\text{-Homology}}$ of Δ-complex

$$H_n^\Delta(X) = \overbrace{\underbrace{\ker(\partial_n)}_{\text{"cycles"}} \big/ \underbrace{\operatorname{im}(\partial_{n+1})}_{\text{"boundaries"}}}^{\text{"classes"}}$$

($\underline{\text{Lemma}}$ says $\operatorname{im}(\partial_{n+1}) \subset \ker(\partial_n)$)

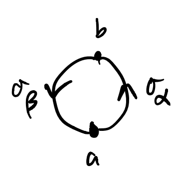
$\underline{\text{Main issue}}$: Not evident $H_n^\Delta(X)$ is $\underline{\text{independent}}$ of Δ-complex str. on X.

$\underline{\text{Ex}}$: $X = S^1$ 

$$\begin{array}{ccc} \Delta_0(X) & \xleftarrow{\partial_1} & \Delta_1(X) \\ \| & & \| \\ \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} \end{array}$$

$0 = \partial_1(\sigma_1)$ 

$H_0^\Delta(X) = \mathbb{Z}$ $\quad H_1^\Delta(X) = \mathbb{Z}$

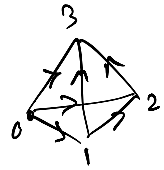
$X = S^1$ 

$$\begin{array}{ccc} \Delta_0(X) & \xleftarrow{\partial_1} & \Delta_1(X) \\ \| & & \| \\ \mathbb{Z}\cdot\sigma_a \oplus \mathbb{Z}\sigma_b & \longleftarrow & \mathbb{Z}\sigma_\alpha \oplus \mathbb{Z}\sigma_\beta \end{array}$$

$$\partial_1 = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}$$

$H_0^\Delta(X) = \mathbb{Z}$ gen by σ_a or σ_b $\qquad H_1^\Delta(X) = \mathbb{Z}$ gen by $\sigma_\alpha + \sigma_\beta$

<u>More exs</u> 1) $X = S^n = \partial \Delta^{n+1}$

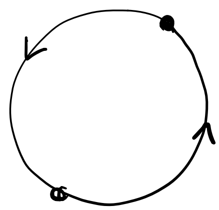


$$H_0^\Delta(X) = \mathbb{Z}, \quad H_1 = \cdots = H_{n-1} = 0, \quad H_n^\Delta = \mathbb{Z}$$

$$X = S^n = \Delta^n \coprod_{\partial \Delta^n} \Delta^n$$

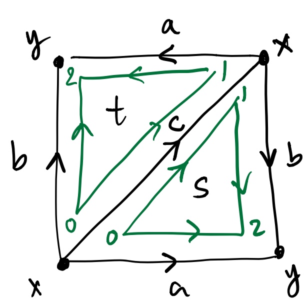
Same answer...

2) $X = \mathbb{RP}^2 =$



$D^2/\sim$

not yet Δ-complex
2-cell is not glued satisfying Δ-complex axioms

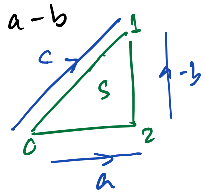


Δ-complex (but not simplicial complex)

$$\mathbb{Z}\cdot x \oplus \mathbb{Z}\cdot y \xleftarrow{\begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}} \mathbb{Z} a \oplus \mathbb{Z}\cdot b \oplus \mathbb{Z} c \xleftarrow{\begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}} \mathbb{Z} s \oplus \mathbb{Z} t$$

gen by x or y $\qquad$ gen by c or $a-b$

$\mathbb{Z} = H_0$ $\qquad$ $\mathbb{Z}/2 = H_1$



$\langle 0 \rangle = H_2$