Another picture of $\pi_1(\tilde{X}, \tilde{x}_0) = <1>$ where $(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0)$ is a universal cover.

When $s = 1$, $\tilde{g}(1) = \tilde{x}_0 = p(\gamma(1)) = p(\gamma(s))$.

Setup: $X$ path-connected, locally path-connected, semi-locally simply-connected.

We've classified path-connected covers $(\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$.

Today: Fix group $G$, classify (not nec path-connected)

$G$-covers $(\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$

Def: A path-connected complex $(X, x_0)$ is called a $K(G, 1)$ if $\pi_1(X, x_0) = G$ and universal cover $(\tilde{X}, \tilde{x}_0)$ is contractible.

Rule: A space $(X, x_0)$ is called a $K(G, n)$ if $\pi_n(X, x_0) = G$ and all other $\pi_i$ trivial (Eilenberg-MacLane spaces).
Thus, any two $K(G, 1)$'s are homotopy equivalent.

**Exs:**
1) $S^1 = K(\mathbb{Z}, 1)$

2) $RP^{\infty} = K(\mathbb{Z}/2, 1)$

3) Closed Riemann surface $\Sigma_g$ of genus $g \geq 1$

$$\cong K(\pi_g, 1) \quad \pi_g = \pi_1(\Sigma_g, x)$$

Recall simply-connected R. surf.: $C, D, \sigma_r^1$

$$\Sigma_g = \begin{cases} C & g=1 \\ D & g \geq 1. \end{cases}$$

4) Knot complements $K \subset S^3$

$$S^3 \setminus K = K(\pi_1, 1)$$

Thus, $K(G, 1)$ exists for any $G$.

**Proof/Construction.** First we'll construct contractible space $EG$ with a free $G$-action. (this will play role of univ. cover.)

**Build out of simplices:**

- $G \xleftarrow{\sigma_0} G \xleftarrow{\sigma_1} G \times G \xleftarrow{\sigma_2} G \times G \times G \xleftarrow{\sigma_3} G \times G \times G \times G \ldots$
- $\Delta^0 \xleftarrow{x} \Delta^1 \xleftarrow{x} \Delta^2 \xleftarrow{x} \Delta^3 \ldots$
Each map $\sigma_i$ tells how to glue ith-boundary face (opposite vertex) to prior simplices: $\sigma_i = \text{forgets ith elt in product.}$

$\mathbb{Z}/2 = \langle 0, 1 \rangle$

$\text{Ex: } G = \mathbb{Z}/2 = \langle 0, 1 \rangle$

![Diagram of a graph with vertices and edges labeled (0,0), (1,0), (0,1), (1,1), with arrows indicating $\sigma_0$ and $\sigma_1$ actions.]

$\sigma_0(0,1) = 1, \sigma_1(0,1) = 0$

give 1-end of $\sigma_0$ to pt $1$

give 0-end of $\sigma_1$ to pt $0$

1-skeleton:

![Graph diagram with vertices (0,0), (1,0), (0,1), (1,1), and edges connecting them.]

(Reyn Construction can be made for any space $X$.)

Lemma: $E_6$ is contractible.

Caution: "Filled in all possible prongs, spheres in space"
**Pf of Lemma:**

A simplex in $EG$ indexed by $(g_0, g_1, \ldots, g_n)$ contracting all points of simplex to vertex of simplex indexed by $(e, g_0, g_1, \ldots, g_n)$ identity.

Thus $EG$ is contracted to 0-simplex indexed by $e$. $\blacksquare$

**Ex:** $G = \mathbb{Z}/2$

- $(0,0)$
- $(0,1)$
- $(1,0)$
- $(1,1)$
- $(0,0)$
- $(0,1)$
- $(0,1,0)$

**Lemma:** $G \& EG$ proper subset by left unit on all factors.

in part, action is free.

**Pf, Exe.** (Hint: $G$ permutes all simplices...)

**Definition:** $BG = EG/\beta$ is a $K(G,1)$.

univ cone $= EG$. 


Then $X$ a pointed CW complex, $Y$ $K(G,1)$

$\text{Hom} \left( \pi_1(X,x_0), G \right) \cong \left\{ (x,x_0), (Y,y_0) \right\}_{x_0} \cong \left\{ \text{G-coverings of } X \text{ with fiber over } x_0 \text{ identified with } G \right\}$

One says $(Y,y_0)$ "classifies" G-coverings + identifications

Given $f : (X,x_0) \rightarrow (Y,y_0)$

$$f_+ : \pi_1(X,x_0) \rightarrow \pi_1(Y,y_0) = G$$

$sp$ homo

$f^* \tilde{Y} = X \times \tilde{Y} \rightarrow X$ G-covering

Cor. Uniqueness (up to homotopy) of $K(G,1)$'s.

Pf. $X,Y$ are $K(G,1)$'s. Then $f$ gives maps (canoc to $id : G \rightarrow G$)

$f : (Y,y_0) \rightarrow (Y,y_0)$, $g : (Y,y_0) \rightarrow (X,x_0)$

Check these are inverse homotopy equivs. □

Let's consider $f : (Y,y_0) \rightarrow (Y,y_0)$

given $\Psi : \pi_1(X,x_0) \rightarrow G$

Under simplifying assumption that $X$ has single 0-cell $x_0$. 
\[X \xrightarrow{f} Y \xrightarrow{K(G,1)} Y\]

\[X^0 \xrightarrow{x_0} X^1 \xrightarrow{e_a} X^2 \xrightarrow{y_0} Y_0\]

\[\psi : \pi_1(Y,y_0) \rightarrow \pi_1(Y_0) \cong G\]

\[\text{clue rep. } \gamma_a\]

\[\text{for } \gamma \in \pi_1(X_0) \subset \pi_1(X) \text{ so } f(\gamma) \in \pi_1(Y) \cong G\]

\[f(e^n) \text{ is } \text{filling limit.}\]

\[n \geq 2 \quad \partial e^n = S^{n-1}\]

\[\text{since } S^{n-1} \text{ has triv } \pi_1, \quad f(e^n) \text{ is } \text{filling limit}\]

\[\text{Uniformly run similar arguments on } X \times I\]