Lecture 1: What is Alg Top?

Def. A top sp $X$ is a set $X$ together with top $\mathcal{T} \subset 2^X = \text{subsets of } X$
1) $\emptyset, X \in \mathcal{T}$
2) $\bigcup_{i \in I} U_i \in \mathcal{T}$ for $U_i \in \mathcal{T}$
3) $\bigcap_{i \in I} U_i \in \mathcal{T}$ for $U_i \in \mathcal{T}$

A map $f : X \rightarrow Y$ is a set map so that $f(U_i) \in \mathcal{T}_Y$ for $U_i \in \mathcal{T}_Y$

Def. Category Top of top sps & maps.

Isms in Top are homeomorphisms.

Top has initial obj $\emptyset$, term obj pt $pt$.

topProd $\cong$ disjoint union, prod $= \text{Cartesian prod} \\ x$

Fully faithful functor $i : \text{Set} \rightarrow \text{Top}$

with ess. image $= \text{spaces with discrete topology}$.

Exer: Do $i^L, i^R$ exist? If yes, what are they?
Find Question: How to classify spaces?

Philosophical role of alg topol: provide a precise language to describe spaces.

Ex. $S^1 \times [0,1] = \text{Cyl}$

orientable

$\text{Mob}$ nonorientable

Def. A top sp $X$ is an $n$-mfd (with bdg.) if locally homeo to $\mathbb{R}^n \ (\mathbb{R}^m \times \mathbb{R}_{\geq 0})$

Ex $\frac{0_+}{0_-} \mathbb{R} \sqcup \mathbb{R} \not\text{ Hausdorff}.$

Mfd may include: Hausdorff, paracompact

provide access to useful tech tools.

Both Cyl, Mob are 2-mfds with bdg.

$\partial \text{Cyl} = S^1 \sqcup S^1 \ , \ \partial \text{Mob} = S^1$

conn comp: 2 $\neq$ 1
Thm. $M^2$ compact 2-mfld classification

orientable

$S^2 \lor T^2 \lor T^2 \lor T^2 \lor (T^2)^g \lor \cdots \lor (T^2)^g$

$\chi = 2 \lor 0 \lor -2 \lor \cdots$

nonorientable

$\mathbb{RP}^2 \lor S^1/\text{antipodal} \lor K^2 \lor \cdots \lor (\mathbb{RP}^2)^k$

$\chi = 1 \lor 0 \lor \cdots$

Euler characteristic distinguish surfaces within each class.

Given space $X$, cut $X$ up into cells (finitely many ...)

$\chi = 1 - \chi + 1 = -2$

$\chi(X) = \sum_{i=0}^{n} (-1)^i (\# \text{ of } i\text{-cells})$

Alg Top: ensure $\chi$ is an invariant!

$\chi(\mathbb{RP}^2) = 1$

$\chi(S^2) = 2$

$\chi(\mathbb{RP}^2) = 1$

$\chi(M^2 \lor N^2) = \chi(M^2) - 1 + \chi(N^2) - 1$
**Def. CW complex**

**Inductive:**
- Base case: \( X_0 = \text{empty set} \) 0 dim.
- Suppose we know what an \( n-1 \) dim CW complex is.

\[
X_n = X_{n-1} \bigcup \frac{1}{\phi_a} \partial D^n_a
\]

\( \phi_a : \partial D^n_a = S^{n-1}_a \to X_{n-1} \)

Attaching \( n-1 \) dim.

\[
X = \bigcup_{n} X_n
\]

**Weak topology**  \( A \subset X \) is qm/cd \( \iff \) \( A \cap X_n \) is qm/cd.

**Closure finite** \( \phi_a (S^{n-1}_a) \) only intersects fin many prior cell interiors.

**Good properties**
1) Hausdorff
2) Locally contractible
3) Products of CW are CW again (at least if countably many cells)

\[\mathbb{R}P^n = \text{lines in } \mathbb{R}^{n+1} \text{ through 0.}\]
\[ X_n = \mathbb{RP}^n = \{ \text{lines in } \mathbb{R}^{n+1} \ [x_0, \ldots, x_n] \} \]

\[ X_{n-1} = \{ \text{lines with } x_n = 0 \} = \mathbb{RP}^{n-1} \]

\[ X_i \setminus X_{i-1} = D_i \]

\[ X_0 = \{ \text{lines with } x_1, \ldots, x_n = 0 \} = \mathbb{RP}^0 \text{ gluing map } \]

\[ \partial D_i = S^{i-1} \to \mathbb{RP}^{i-1} \]