Lecture 1 What is Alg Top?

Def A top sp X is a set X
together with top $\tau \subset 2^X$ = subsets of X

1) $\emptyset, X \in \tau$
2) $\bigcup_{\substack{i \in I \\ \text{arbitrary}}} \mathcal{U}_i \in \tau$ for $\mathcal{U}_i \in \tau$
3) $\bigcap_{\substack{i \in I \\ \text{finitely many}}} \mathcal{U}_i \in \tau$ for $\mathcal{U}_i \in \tau$

A map $f : X \to Y$ is a set map
so that $f^{-1}(\mathcal{U}_i) \in \tau_X$, for $\mathcal{U}_i \in \tau_Y$

Def. Category Top of top sps & maps.

Isms in Top are homeomorphisms

Top has initial obj $\emptyset$, term obj pt

coprod = disj union $\amalg$, prod = Cartesian prod $\times$

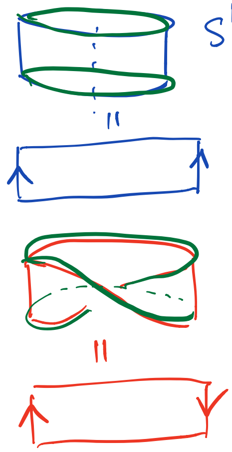
Fully faithful functor $i : \text{Set} \to \text{Top}$
with ess. image = spaces with discrete topol.

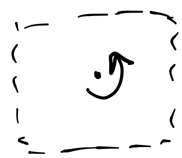
Exer Do i^L, i^R exist? If yes, what are they?

Fund Question: How to classify spaces?

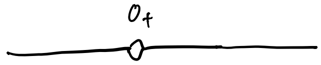
Philosophical role of alg topol: provide a precise language to describe spaces.

Ex.





Def. A top sp X is an n-mfd (with bdy) if locally homeo to $\mathbb{R}^n$ $(\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0})$

Ex 

$\mathbb{R} \sqcup_{\mathbb{R}_{\neq 0}} \mathbb{R}$ not Hausdorff.

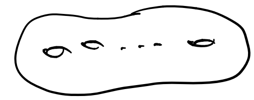
Mfd may include: Hausdorff, paracompact
provide access to useful tech. tools.

Both Cyl, Mob are 2-mfds with bdy.

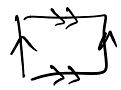
$\partial Cyl = S^1 \sqcup S^1$, $\partial Mob = S^1$

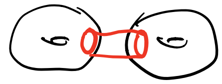
conn comps: $2 \neq 1$

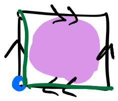
Thm M^2 compact 2-mfd classification

S^2 , T^2 , , ..., $(T^2)^{\# g}$ 

orientable $\chi = 2$ $= S^1 \times S^1$ $\chi = 0$ $T^2 \# T^2$ $\chi = -2$ ...



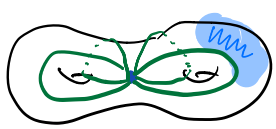
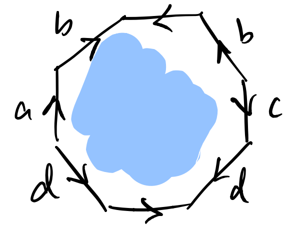


nonorientable $\mathbb{RP}^2 = S^2/\text{antipodal}$, K^2 , ..., $(\mathbb{RP}^2)^{\# k}$

$\chi = 1$ $= \mathbb{RP}^2 \# \mathbb{RP}^2$ $\chi = 0$ ...

Euler characteristic distinguish surfaces within each class.

Given space X, cut X up into cells (finitely many...)

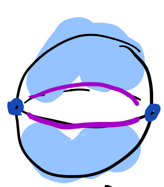
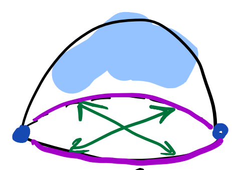
 = 

$\chi = 1 - 4 + 1$
$= -2$

$$\chi(X) = \sum_{i=0}^{n} (-1)^i (\# \text{ of } i\text{-cells})$$

Alg Top: ensure χ is an invariant!

Ex $\chi(\mathbb{RP}^2) = 1$

 $\xrightarrow{2 \to 1}$ 

S^2 $\chi = 2$ $\mathbb{RP}^2$ $\chi = 1$

Exer $\chi(M^2 \# N^2) = \chi(M^2) - 1 + \chi(N^2) - 1$

Def CW complex

Inductive:

Base case X_0 = discrete set 0 dim

Suppose we know what an $n-1$ dim CW complex is

$$X_n = X_{n-1} \coprod_{\varphi_\alpha} D^n_\alpha$$

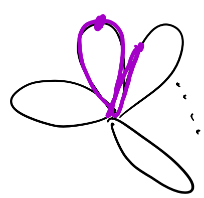
$\alpha \in A$

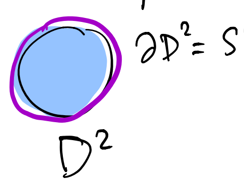
$\varphi_\alpha : \partial D^n_\alpha = S^{n-1}_\alpha \longrightarrow X_{n-1}$
attaching map

$$X = \bigcup_n X_n$$

Weak topology $A \subset X$ is open/cl $\Leftrightarrow$ $A \cap X_n$ is open/cl.

Closure finite $\varphi_\alpha(S^{n-1}_\alpha)$ only intersects fin many prior cell interiors

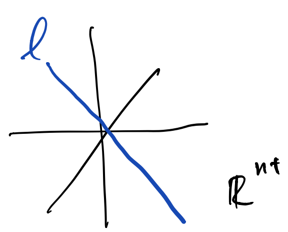




Good properties
1) Hausdorff
2) locally contractible
3) products of CW are CW again (at least if countably many cells)

⋮

Ex $\mathbb{R}P^n$ = lines in $\mathbb{R}^{n+1}$ through 0.



$X_n = \mathbb{RP}^n = \{ \text{lines in } \mathbb{R}^{n+1} \ [x_0, \ldots, x_n] \}$

$\cup$

$X_{n-1} = \{ \text{lines with } x_n = 0 \} = \mathbb{RP}^{n-1}$

$X_i \setminus X_{i-1} = \mathring{D}^i$

$\cup$

$\vdots \qquad \vdots$

$\cup$

$X_0 = \{ \text{lines with } x_1, \ldots, x_n = 0 \} = \mathbb{RP}^0$

~~Exer~~ Describe gluing map

$\partial D^i = S^{i-1} \longrightarrow \mathbb{RP}^{i-1}$