Name: ______________ Math 256A Midterm Due Thursday 10/24

1. Let $C_0(\mathbb{C})$ be the $\mathbb{C}$-algebra of complex-valued continuous functions on $\mathbb{C}$ with its classical topology.
   (a) Describe the maximal ideals of $C_0(\mathbb{C})$. Is (0) a prime ideal of $C_0(\mathbb{C})$?
   (b) Calculate the stalk of the structure sheaf of $\operatorname{Spec} C_0(\mathbb{C})$ at the maximal ideals.
   (c) Determine if the underlying topological space of $\operatorname{Spec} C_0(\mathbb{C})$ is i) connected, ii) quasi-compact, iii) irreducible, iv) Noetherian.
   (d) For any prime ideal $\mathfrak{p} \subset C_0(\mathbb{C})$, calculate $V(\mathfrak{p}) \subset \operatorname{Spec} C_0(\mathbb{C})$.

2. Let X and Y be schemes and R a commutative ring.
   (a) Show scheme maps $X \to \operatorname{Spec} R$ are in natural bijection with ring maps $R \to \Gamma(X, \mathcal{O}_X)$. Construct a left adjoint to the inclusion AffineSchemes $\to$ Schemes.
   (b) Show any scheme map $\operatorname{Spec} R \to X$ induces a ring map $\Gamma(X, \mathcal{O}_X) \to R$ but show by example this does not give a bijection.
   (c) Explain precisely how a scheme map $f : X \to Y$ with $f(x) = y$ induces a map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ on the stalks of the structure sheaves.
   (d) True or false: any scheme admits an open cover by affines such that the intersections of pairs of opens in the cover are also affine. If true, prove; if false, give a counterexample.

3. Let $GL(n, \mathbb{C})$ be the group of $(n \times n)$-invertible complex matrices. Let $Gr(k, n, \mathbb{C})$ be the set of k-dimensional subspaces of $\mathbb{C}^n$.
   (a) Find a finitely-generated, integral $\mathbb{C}$-algebra $R(n)$ such that $GL(n) := \operatorname{Spec} R(n)$ has closed points $GL(n, \mathbb{C})$. Equip $GL(n)$ with a group-scheme structure inducing the usual group structure on $GL(n, \mathbb{C})$.
   (b) Find a finitely-generated, $\mathbb{Z}_{\geq 0}$-graded, integral $\mathbb{C}$-algebra $R(k, n)$ such that $Gr(k, n) := \operatorname{Proj} R(k, n)$ has closed points $Gr(k, n, \mathbb{C})$. Equip $Gr(k, n)$ with an action of $GL(n)$ inducing the usual action of $GL(n, \mathbb{C})$ on $Gr(k, n, \mathbb{C})$.
   (Your construction should specialize to $R(1, n) = \mathbb{C}[x_0, \ldots, x_n]$ with $|x_i| = 1$ for $i = 0, \ldots, n$.)
   (c) Fix a plane $P \in Gr(2, 4, \mathbb{C})$. Construct an open subscheme $U \subset Gr(2, 4)$ whose closed points are those planes $Q \in Gr(2, 4, \mathbb{C})$ with $\dim(P \cap Q) = 0$. Is U a familiar scheme?
   (d) Fix another plane $P' \in Gr(2, 4, \mathbb{C})$ such that $\dim(P \cap P') = 0$. Construct a reduced subscheme $X \subset Gr(2, 4)$ whose closed points are those planes $Q \in Gr(2, 4, \mathbb{C})$ with $\dim(P \cap Q) \geq 1$ and $\dim(P' \cap Q) = 0$. Describe X as a hypersurface in $\mathbb{A}^4$.

4. Consider the projective plane $\mathbb{P}^2 = \operatorname{Proj} \mathbb{C}[x, y, z]$. Recall that conics $C \subset \mathbb{P}^2$ (possibly degenerate) form a projective space $\mathbb{P}^5 = \operatorname{Proj} \mathbb{C}[A, B, C, D, E, F]$.
   (a) Fix a line $\mathbb{P}^1 \simeq \ell \subset \mathbb{P}^2$. Describe the subscheme $X_{1,0} \subset \mathbb{P}^5$ of conics $C \subset \mathbb{P}^2$ tangent to ℓ.
   (You can take the definition of tangent to be that the topological space underlying the intersection $C \times_{\mathbb{P}^2} \ell$ is a single point.)
   (b) Describe the subscheme $X_{1,4} \subset \mathbb{P}^5$ of conics $C \subset \mathbb{P}^2$ tangent to ℓ and passing through 4 generic points.
   (c) Fix a distinct line $\mathbb{P}^1 \simeq \ell' \neq \ell \subset \mathbb{P}^2$. Describe the subscheme $X_{2,3} \subset \mathbb{P}^5$ of conics $C \subset \mathbb{P}^2$ tangent to ℓ and ℓ' and passing through 3 generic points.
   (d) Let $X_{m,n} \subset \mathbb{P}^5$ be the scheme of conics passing through m generic points and tangent to n generic lines. Can you find a relationship between $X_{m,n}$ and $X_{n,m}$?