

Choose one of the following topics. Write an exposition providing answers to the questions, along with supporting arguments, examples, and references. Conclude with a discussion of what you would like to study next in the chosen direction.

You may work over the complex numbers or more generally if you prefer.

1. How is the group structure on an elliptic curve C related to the group structure on $\text{Pic}(C)$? Classify semistable degree 0 algebraic vector bundles $E \rightarrow C$ on an elliptic curve. What happens if we consider a singular plane cubic with a node or cusp?

(Elliptic curve = smooth projective complex curve of genus 1 with marked point. Degree = degree of determinant line bundle; semistable = no vector subbundle of positive degree.)

2. Define the notion of smooth resolution $\tilde{X} \rightarrow X$ of a scheme. Compare methods of finding a smooth resolution of a curve: Newton's method, normalization, valuations, blowing up, or any you find interesting. Describe a method of finding smooth resolutions of surfaces, and run it on the type A rational singularities $xy - z^n = 0$, $n \geq 2$.
3. Are all smooth and proper curves projective? Describe which line bundles are very ample. Give examples of a proper non-projective surface and a proper smooth non-projective three-fold. What obstructs your examples from having very ample line bundles?