Name:______________ Math 261A Midterm 2

All representations are assumed to be finite-dimensional and complex unless otherwise stated

1. (10 points) State whether each assertion is always true (T) or sometimes false (F).
   (a) (1 point) __F__ Every representation of the Lie algebra $\mathbb{C}$ is a direct sum of irreducibles.
   (b) (1 point) __F__ Every representation of the Lie algebra $\mathfrak{gl}(2,\mathbb{C})$ is a direct sum of irreducibles.
   (c) (1 point) __T__ The Killing form on $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$ is non-degenerate.
   (d) (1 point) __F__ The Killing form on $\mathfrak{b}(2,\mathbb{C}) = \{A \in \mathfrak{sl}(2,\mathbb{C}) \mid A \text{ upper triangular}\}$ is non-degenerate.
   (e) (1 point) __F__ Every irreducible representation of $\mathfrak{n}(2,\mathbb{C}) = \{A \in \mathfrak{sl}(2,\mathbb{C}) \mid A \text{ strictly upper triangular}\}$ is isomorphic to the trivial representation $\mathbb{C}$.
   (f) (1 point) __F__ Every representation of $\mathfrak{so}(3,\mathbb{C})$ has an invariant non-degenerate inner product.
   (g) (1 point) __T__ Every representation of $SO(3,\mathbb{C})$ has an invariant non-degenerate inner product.
   (h) (1 point) __F__ For every finite-dimensional complex Lie algebra $\mathfrak{g}$, there exists a compact Lie group $G_c$ with Lie algebra $\mathfrak{g}_c$ satisfying $\mathfrak{g}_c \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{g}$.
   (i) (1 point) __T__ For every connected Lie group $G$, its category of representations is equivalent to a full subcategory of that of its Lie algebra $\mathfrak{g}$.
   (j) (1 point) __F__ There exist non-trivial connected Lie groups whose finite-dimensional representation are all trivial.

2. (10 points) Let $V_1 = \mathbb{C}^2$ denote the standard representation of $\mathfrak{sl}(2,\mathbb{C})$, $V_n = \mathrm{Sym}^n(V_1)$ its $n$-fold symmetric product, and $V_1^{\otimes n} = V_1 \otimes \cdots V_1$ its $n$-fold tensor product. Let $\mathbb{C}[e_n \mid n \in \mathbb{Z}]$ denote the ring of formal characters (weight multiplicities).
   (a) (2 points) What is the formal character of $V_n$?

   (a) $\sum_{k=0}^{n} e_{-n+2k} = e_n + e_{n-2} + \cdots + e_{-n}$

   (b) (2 points) What is the formal character of $V_1^{\otimes n}$?

   (b) $(e_1 + e_{-1})^n = \sum_{k=0}^{n} \binom{n}{k} e_{-n+2k}$

   (c) (2 points) What is the multiplicity of the trivial representation $V_0$ in $V_1^{\otimes 2n}$?

   (c) $\binom{2n}{n} - \binom{2n}{n-1}$

   (d) (2 points) What is the multiplicity of the irreducible representation $V_{n-2}$ in $V_1^{\otimes n}$?

   (d) $n-1$

   (e) (2 points) What is the dimension of $\mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(V_1^{\otimes n}, V_1^{\otimes n})$?

   (e) $\binom{2n}{n} - \binom{2n}{n-1}$

3. (10 points) (a) (5 points) Partition the following Lie algebras into classes so that those within each class are isomorphic to each other and those in different classes are not isomorphic.

   $\mathfrak{sl}(2,\mathbb{C})$ $\mathfrak{sl}(3,\mathbb{C})$ $\mathfrak{sl}(4,\mathbb{C})$ $\mathfrak{so}(3,\mathbb{C})$ $\mathfrak{so}(4,\mathbb{C})$ $\mathfrak{so}(5,\mathbb{C})$ $\mathfrak{so}(6,\mathbb{C})$ $\mathfrak{sp}(2,\mathbb{C})$ $\mathfrak{sp}(4,\mathbb{C})$ $\mathfrak{sp}(6,\mathbb{C})$

**Solution:** 1) $\mathfrak{sl}(2,\mathbb{C})$ $\mathfrak{so}(3,\mathbb{C})$ $\mathfrak{sp}(2,\mathbb{C})$
2) $\mathfrak{sl}(3,\mathbb{C})$
3) $\mathfrak{sl}(4,\mathbb{C})$ $\mathfrak{so}(6,\mathbb{C})$
4) $\mathfrak{so}(4,\mathbb{C})$
5) $\mathfrak{so}(5,\mathbb{C})$ $\mathfrak{sp}(4,\mathbb{C})$
6) $\mathfrak{sp}(6,\mathbb{C})$

(b) (5 points) Partition the following Lie groups into classes so that those within each class have isomorphic Lie algebras and those in different classes have Lie algebras that are not isomorphic. ($U(1,1)$ denotes the subgroup of $GL(2,\mathbb{C})$ preserving the indefinite hermitian inner product $|z_1|^2 - |z_2|^2$; $GL(1,\mathbb{H})$ denotes the group of non-zero quaternions.)

$$U(1)\times U(1) \quad U(1,1) \quad U(2) \quad SU(1,1)\times U(1) \quad SU(2)\times U(1)$$

$$O(3,\mathbb{R})\times O(2,\mathbb{R}) \quad SO(3,\mathbb{R})\times SO(2,\mathbb{R}) \quad SL(2,\mathbb{R})\times \mathbb{R}^\times \quad GL(2,\mathbb{R}) \quad GL(1,\mathbb{H})$$

**Solution:** 1) $U(1)\times U(1)$
2) $U(1,1)$ $SU(1,1)\times U(1)$ $SL(2,\mathbb{R})\times\mathbb{R}^\times$ $GL(2,\mathbb{R})$
2) $U(2)$ $SU(2)\times U(1)$ $O(3,\mathbb{R})\times O(2,\mathbb{R})$ $SO(3,\mathbb{R})\times SO(2,\mathbb{R})$ $GL(1,\mathbb{H})$

4. (14 points) Consider the natural action of $SL(2,\mathbb{C})$ on $\mathbb{CP}^1$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}), \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{CP}^1$$

and the induced Lie algebra map $\mathfrak{sl}(2,\mathbb{C}) \to \mathrm{Vect}(\mathbb{CP}^1)$.

Let $\mathbb{C} = \{x \neq 0\} \subset \mathbb{CP}^1$ be the complex line where the slope $s = y/x$ is well-defined.

(a) (2 points) Calculate the the image of

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

under the induced map $\mathfrak{sl}(2,\mathbb{C}) \to \mathrm{Vect}(\mathbb{C})$

(a) $\partial_s$

Consider the ring of polynomial functions $\mathbb{C}[s]$ as an infinite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-representation.

(b) (2 points) Calculate the formal character of $\mathbb{C}[s]$.

(b) $\sum_{n=0}^{\infty} e_{2n}$

(c) (2 points) List the irreducible subrepresentations of $\mathbb{C}[s]$.

(c) **trivial** $V_0 \simeq \mathbb{C}\cdot 1$

(d) (2 points) List the irreducible quotient representations of $\mathbb{C}[s]$.

(d) **Verma for $\mathfrak{b}^{op}$ with lowest weight 2**

Let $\mathbb{C}^\times = \{x, y \neq 0\} \subset \mathbb{CP}^1$ be the punctured complex line where the slope $s = y/x$ and inverse slope $s^{-1} = x/y$ are both well-defined. Consider its ring of polynomial functions $\mathbb{C}[s, s^{-1}]$ as an infinite-dimensional $\mathfrak{sl}(2, \mathbb{C})$-representation.

(e) (2 points) Calculate the formal character of $\mathbb{C}[s, s^{-1}]$.

(e) $\sum_{n=-\infty}^{\infty} e_{-2n}$

(f) (2 points) List the irreducible subrepresentations of $\mathbb{C}[s, s^{-1}]$.

(f) **trivial** $V_0 \simeq \mathbb{C} \cdot 1$

(g) (2 points) List the irreducible quotient representations of $\mathbb{C}[s, s^{-1}]$.

(g) **Verma for $\mathfrak{b}$ with highest weight $-2$, Verma for $\mathfrak{b}^{op}$ with lowest weight 2**

5. (6 points) Consider the adjoint representation of $\mathfrak{sl}(3, \mathbb{C})$. For each listed representation of $\mathfrak{sl}(3, \mathbb{C})$, calculate the dimension of its subspace of invariants (multiplicity of trivial representation as a summand).

(a) (2 points) $\mathrm{Sym}^2(\mathfrak{sl}(3, \mathbb{C}))$?

(a) **1**

(b) (2 points) $\Lambda^2(\mathfrak{sl}(3, \mathbb{C}))$?

(b) **0**

(c) (2 points) $\Lambda^3(\mathfrak{sl}(3, \mathbb{C}))$?

(c) **1**