Name:______________ Math 261A Midterm 2

All representations are assumed to be finite-dimensional and complex unless otherwise stated

1. (10 points) State whether each assertion is always true (T) or sometimes false (F).
   (a) (1 point) ____ Every representation of the Lie algebra $\mathbb{C}$ is a direct sum of irreducibles.
   (b) (1 point) ____ Every representation of the Lie algebra $\mathfrak{gl}(2,\mathbb{C})$ is a direct sum of irreducibles.
   (c) (1 point) ____ The Killing form on $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$ is non-degenerate.
   (d) (1 point) ____ The Killing form on $\mathfrak{b}(2,\mathbb{C}) = \{A \in \mathfrak{sl}(2,\mathbb{C}) \mid A \text{ upper triangular}\}$ is non-degenerate.
   (e) (1 point) ____ Every irreducible representation of $\mathfrak{n}(2,\mathbb{C}) = \{A \in \mathfrak{sl}(2,\mathbb{C}) \mid A \text{ strictly upper triangular}\}$ is isomorphic to the trivial representation $\mathbb{C}$.
   (f) (1 point) ____ Every representation of $\mathfrak{so}(3,\mathbb{C})$ has an invariant non-degenerate inner product.
   (g) (1 point) ____ Every representation of $SO(3,\mathbb{C})$ has an invariant non-degenerate inner product.
   (h) (1 point) ____ For every finite-dimensional complex Lie algebra $\mathfrak{g}$, there exists a compact Lie group $G_c$ with Lie algebra $\mathfrak{g}_c$ satisfying $\mathfrak{g}_c \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{g}$.
   (i) (1 point) ____ For every connected Lie group $G$, its category of representations is equivalent to a full subcategory of that of its Lie algebra $\mathfrak{g}$.
   (j) (1 point) ____ There exist non-trivial connected Lie groups whose finite-dimensional representation are all trivial.

2. (10 points) Let $V_1 = \mathbb{C}^2$ denote the standard representation of $\mathfrak{sl}(2,\mathbb{C})$, $V_n = \mathrm{Sym}^n(V_1)$ its $n$-fold symmetric product, and $V_1^{\otimes n} = V_1 \otimes \cdots V_1$ its $n$-fold tensor product. Let $\mathbb{C}[e_n \mid n \in \mathbb{Z}]$ denote the ring of formal characters (weight multiplicities).
   (a) (2 points) What is the formal character of $V_n$?

   (a) ______________________________

   (b) (2 points) What is the formal character of $V_1^{\otimes n}$?

   (b) ______________________________

   (c) (2 points) What is the multiplicity of the trivial representation $V_0$ in $V_1^{\otimes 2n}$?

   (c) ______________________________

   (d) (2 points) What is the multiplicity of the irreducible representation $V_{n-2}$ in $V_1^{\otimes n}$?

   (d) ______________________________

   (e) (2 points) What is the dimension of $\mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(V_1^{\otimes n}, V_1^{\otimes n})$?

   (e) ______________________________

3. (10 points) (a) (5 points) Partition the following Lie algebras into classes so that those within each class are isomorphic to each other and those in different classes are not isomorphic.

   $\mathfrak{sl}(2,\mathbb{C})$ $\mathfrak{sl}(3,\mathbb{C})$ $\mathfrak{sl}(4,\mathbb{C})$ $\mathfrak{so}(3,\mathbb{C})$ $\mathfrak{so}(4,\mathbb{C})$ $\mathfrak{so}(5,\mathbb{C})$ $\mathfrak{so}(6,\mathbb{C})$ $\mathfrak{sp}(2,\mathbb{C})$ $\mathfrak{sp}(4,\mathbb{C})$ $\mathfrak{sp}(6,\mathbb{C})$

(b) (5 points) Partition the following Lie groups into classes so that those within each class have isomorphic Lie algebras and those in different classes have Lie algebras that are not isomorphic. ($U(1,1)$ denotes the subgroup of $GL(2,\mathbb{C})$ preserving the indefinite hermitian inner product $|z_1|^2 - |z_2|^2$; $GL(1,\mathbb{H})$ denotes the group of non-zero quaternions.)

$$U(1)\times U(1) \quad U(1,1) \quad U(2) \quad SU(1,1)\times U(1) \quad SU(2)\times U(1)$$

$$O(3,\mathbb{R})\times O(2,\mathbb{R}) \quad SO(3,\mathbb{R})\times SO(2,\mathbb{R}) \quad SL(2,\mathbb{R})\times \mathbb{R}^\times \quad GL(2,\mathbb{R}) \quad GL(1,\mathbb{H})$$

4. (14 points) Consider the natural action of $SL(2,\mathbb{C})$ on $\mathbb{CP}^1$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}), \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{CP}^1$$

and the induced Lie algebra map $\mathfrak{sl}(2,\mathbb{C}) \to \mathrm{Vect}(\mathbb{CP}^1)$.

Let $\mathbb{C} = \{x \neq 0\} \subset \mathbb{CP}^1$ be the complex line where the slope $s = y/x$ is well-defined.

(a) (2 points) Calculate the the image of

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

under the induced map $\mathfrak{sl}(2,\mathbb{C}) \to \mathrm{Vect}(\mathbb{C})$

(a) ________________________________________

Consider the ring of polynomial functions $\mathbb{C}[s]$ as an infinite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-representation.

(b) (2 points) Calculate the formal character of $\mathbb{C}[s]$.

(b) ________________________________________

(c) (2 points) List the irreducible subrepresentations of $\mathbb{C}[s]$.

(c) ________________________________________

(d) (2 points) List the irreducible quotient representations of $\mathbb{C}[s]$.

(d) ________________________________________

Let $\mathbb{C}^\times = \{x, y \neq 0\} \subset \mathbb{CP}^1$ be the punctured complex line where the slope $s = y/x$ and inverse slope $s^{-1} = x/y$ are both well-defined. Consider its ring of polynomial functions $\mathbb{C}[s, s^{-1}]$ as an infinite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-representation.

(e) (2 points) Calculate the formal character of $\mathbb{C}[s, s^{-1}]$.

(e) ______________________________

(f) (2 points) List the irreducible subrepresentations of $\mathbb{C}[s, s^{-1}]$.

(f) ______________________________

(g) (2 points) List the irreducible quotient representations of $\mathbb{C}[s, s^{-1}]$.

(g) ______________________________

5. (6 points) Consider the adjoint representation of $\mathfrak{sl}(3, \mathbb{C})$. For each listed representation of $\mathfrak{sl}(3, \mathbb{C})$, calculate the dimension of its subspace of invariants (multiplicity of trivial representation as a summand).

(a) (2 points) $\mathrm{Sym}^2(\mathfrak{sl}(3, \mathbb{C}))$?

(a) ______________________________

(b) (2 points) $\Lambda^2(\mathfrak{sl}(3, \mathbb{C}))$?

(b) ______________________________

(c) (2 points) $\Lambda^3(\mathfrak{sl}(3, \mathbb{C}))$?

(c) ______________________________