SID:______________ Math 261A Midterm 1 Grader SID:______________

1. (10 points) State whether each assertion is always true (T) or sometimes false (F).
   (a) (1 point) __**F**__ The set of connected components $\pi_0(G)$ of a Lie group is an abelian group.
   (b) (1 point) __**T**__ The fundamental group $\pi_1(G, e)$ of a Lie group is an abelian group.
   (c) (1 point) __**T**__ A normal discrete subgroup $\Gamma \subset G$ of a connected Lie group is abelian.
   (d) (1 point) __**F**__ The universal cover of $\mathrm{SL}(2, \mathbb{C})$ is contractible.
   (e) (1 point) __**T**__ The universal cover of $\mathrm{SL}(2, \mathbb{R})$ is contractible.
   (f) (1 point) __**T**__ $\mathfrak{sl}(2, \mathbb{C}) \simeq \mathfrak{so}(3, \mathbb{C})$.
   (g) (1 point) __**F**__ $\mathfrak{sl}(2, \mathbb{R}) \simeq \mathfrak{so}(3, \mathbb{R})$.
   (h) (1 point) __**F**__ If $G$ acts transitively on $X$, then the natural map $\mathfrak{g} \to \mathrm{Vect}(X)$ is surjective.
   (i) (1 point) __**T**__ If $G$ acts freely on $X$, then the natural map $\mathfrak{g} \to \mathrm{Vect}(X)$ is injective.
   (j) (1 point) __**F**__ If $G$ acts on $X$, and the natural map $\mathfrak{g} \to \mathrm{Vect}(X)$ is injective, then $G$ acts freely.

2. (10 points) Let $\mathrm{SL}(2, \mathbb{C})$ be the Lie group of $2 \times 2$ complex matrices of determinant 1.
   (a) (2 points) Describe the matrices in the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.

   (a) $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, a, b, c, d \in \mathbb{C}$

   Consider the element

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C})$$

   (b) (2 points) Find the eigenvalues of the operator $ad_H : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{sl}(2, \mathbb{C})$.

   (b) $-2, 0, 2$

   (c) (2 points) Find a basis of corresponding eigenvectors.

   (c) $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

   (d) (2 points) Calculate the Killing form pairing $\langle H, H \rangle_K$.

   (d) 8

   (e) (2 points) Calculate the matrix of the Killing form with respect to your basis.

   (e) $\begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}$

3. (10 points) (a) (2 points) State the Jacobi identity.

   (a) $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$

   Let $\mathfrak{g}$ be a Lie algebra. Define the Lie ideal $[\mathfrak{g}, \mathfrak{g}] = span\langle [v, w] \in \mathfrak{g} \,|\, v, w \in \mathfrak{g} \rangle$.
   Calculate the Lie algebra $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ for the following.

   (b) (2 points) $\mathfrak{gl}(2, \mathbb{C}) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{C}\}$

   (b) $\mathbb{C}$

(c) (2 points) $\mathfrak{sl}(2,\mathbb{C}) = \{A \in \mathfrak{gl}(2,\mathbb{C}), tr(A) = 0\}$

(c) $\langle 0 \rangle$

(d) (2 points) $\mathfrak{b} = \{A \in \mathfrak{gl}(2,\mathbb{C})$ upper triangular$\}$

(d) $\mathbb{C} \oplus \mathbb{C}$

(e) (2 points) $\mathrm{Vect}(\mathbb{R}) = \{$vector fields on $\mathbb{R}\}$

(e) $\langle 0 \rangle$

4. (10 points) Let $\mathrm{SL}(2,\mathbb{R})$ be the Lie group of $2 \times 2$ real matrices of determinant 1. Let $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. Consider the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{CP}^1$ by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

(a) (2 points) List the orbits.

(a) $\mathbb{RP}^1, \mathbb{H}^+, \mathbb{H}^-$

(b) (2 points) What is the stabilizer of $z = 0$?

(b) $\{\begin{pmatrix} r & u \\ 0 & r^{-1} \end{pmatrix}\}, r \in \mathbb{R}^\times, u \in \mathbb{R}$

(c) (2 points) What is the stabilizer of $z = i$?

(c) $\{\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\}, \theta \in \mathbb{R}/2\pi$

(d) (2 points) Calculate the image $\tilde{v} \in \mathrm{Vect}(\mathbb{CP}^1)$ of the vector $v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R})$ under the infinitesimal action $\mathfrak{sl}(2,\mathbb{R}) \to \mathrm{Vect}(\mathbb{CP}^1)$.

(d) $\partial_z$

(e) (2 points) Calculate the image $\tilde{v}_i \in T_0\mathbb{CP}^1$ of the vector $v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R})$ under the restriction of the infinitesimal action $\mathfrak{sl}(2,\mathbb{R}) \to T_0\mathbb{CP}^1$ to $0 \in \mathbb{CP}^1$.

(e) $0$

5. (10 points) Let $G$ be a Lie group acting on a manifold $X$, and $\mathfrak{g} \to \mathrm{Vect}(X), v \mapsto \tilde{v}$ the corresponding infinitesimal action. Define the moment map

$$\mu : T^*X \longrightarrow \mathfrak{g}^* \qquad \langle \mu(x,\xi), v \rangle = \xi(\tilde{v}_x) \qquad v \in \mathfrak{g}, \xi \in T_x^*X$$

Calculate $\mu$ in the following cases using the identification $T^*\mathbb{R}^n \simeq \mathbb{R}^n \times (\mathbb{R}^n)^*$ to write the moment map in the form $\mu(x,\xi)$, for $x \in \mathbb{R}^n, \xi \in (\mathbb{R}^n)^*$.

(a) (2 points) Standard action $r \cdot x = rx$ of $G = \mathrm{GL}(1,\mathbb{R})$ on $X = \mathbb{R}$.

(a) $\mu_v(x,\xi) = x\xi$

(b) (2 points) Hyperbolic action $r \cdot (x_1, x_2) = (rx_1, r^{-1}x_2)$ of $G = \mathrm{GL}(1,\mathbb{R})$ on $X = \mathbb{R}^2$.

(b) $\mu_v(x,\xi) = x_1\xi_1 - x_2\xi_2$

For the following cases, use the identification $T^*G \simeq G \times \mathfrak{g}^*$ induced by the identification $TG \simeq G \times \mathfrak{g}$ given by right-invariant vector fields to write the moment map in the form $\mu(g,\xi)$, for $g \in G, \xi \in \mathfrak{g}^*$.

(c) (2 points) Trivial action of $G$ on itself.

(c) $\mu(g, \xi) = 0$

(d) (2 points) Left multiplication action of $G$ on itself.

(d) $\mu(g, \lambda) = \xi$

(e) (2 points) Right multiplication action of $G$ on itself.

(e) $\mu(g, \lambda) = Ad_g(\xi)$