1. (10 points) State whether each assertion is always true (T) or sometimes false (F).

(a) (1 point) ____ The set of connected components \( \pi_0(G) \) of a Lie group is an abelian group.
(b) (1 point) ____ The fundamental group \( \pi_1(G,e) \) of a Lie group is an abelian group.
(c) (1 point) ____ A normal discrete subgroup \( \Gamma \subset G \) of a connected Lie group is abelian.
(d) (1 point) ____ The universal cover of \( \text{SL}(2,\mathbb{C}) \) is contractible.
(e) (1 point) ____ The universal cover of \( \text{SL}(2,\mathbb{R}) \) is contractible.
(f) (1 point) ____ \( \mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{so}(3,\mathbb{C}) \).
(g) (1 point) ____ \( \mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{so}(3,\mathbb{R}) \).
(h) (1 point) ____ If \( G \) acts transitively on \( X \), then the natural map \( g \to \text{Vect}(X) \) is surjective.
(i) (1 point) ____ If \( G \) acts freely on \( X \), then the natural map \( g \to \text{Vect}(X) \) is injective.
(j) (1 point) ____ If \( G \) acts on \( X \), and the natural map \( g \to \text{Vect}(X) \) is injective, then \( G \) acts freely.

2. (10 points) Let \( \text{SL}(2,\mathbb{C}) \) be the Lie group of \( 2 \times 2 \) complex matrices of determinant 1.

(a) (2 points) Describe the matrices in the Lie algebra \( \mathfrak{sl}(2,\mathbb{C}) \).

Consider the element
\[
H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C})
\]

(b) (2 points) Find the eigenvalues of the operator \( \text{ad}_H : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{sl}(2,\mathbb{C}) \).

(c) (2 points) Find a basis of corresponding eigenvectors.

(d) (2 points) Calculate the Killing form pairing \( \langle H, H \rangle_K \).

(e) (2 points) Calculate the matrix of the Killing form with respect to your basis.

3. (10 points) (a) (2 points) State the Jacobi identity.

Let \( \mathfrak{g} \) be a Lie algebra. Define the Lie ideal \( [\mathfrak{g}, \mathfrak{g}] = \text{span}(v,w) \in \mathfrak{g} \mid v,w \in \mathfrak{g} \).

Calculate the Lie algebra \( \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \) for the following.

(b) (2 points) \( \mathfrak{gl}(2,\mathbb{C}) = \{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{C} \} \)

(c) (2 points) \( \mathfrak{sl}(2,\mathbb{C}) = \{ A \in \mathfrak{gl}(2,\mathbb{C}), \text{tr}(A) = 0 \} \)
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(d) (2 points) \( b = \{ A \in \mathfrak{gl}(2, \mathbb{C}) \text{ upper triangular} \} \)

(e) (2 points) \( \text{Vect}(\mathbb{R}) = \{ \text{vector fields on } \mathbb{R} \} \)

4. (10 points) Let \( \text{SL}(2, \mathbb{R}) \) be the Lie group of \( 2 \times 2 \) real matrices of determinant 1. Let \( \mathbb{CP}^1 = \mathbb{C} \cup \{ \infty \} \) be the Riemann sphere. Consider the action of \( \text{SL}(2, \mathbb{R}) \) on \( \mathbb{CP}^1 \) by fractional linear transformations

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}
\]

(a) (2 points) List the orbits.

(b) (2 points) What is the stabilizer of \( z = 0 \)?

(c) (2 points) What is the stabilizer of \( z = i \)?

(d) (2 points) Calculate the image \( \tilde{v} \in \text{Vect}(\mathbb{CP}^1) \) of the vector \( v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}) \) under the infinitesimal action \( \mathfrak{sl}(2, \mathbb{R}) \to \text{Vect}(\mathbb{CP}^1) \).

(e) (2 points) Calculate the image \( \tilde{v}_i \in T_0\mathbb{CP}^1 \) of the vector \( v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}) \) under the restriction of the infinitesimal action \( \mathfrak{sl}(2, \mathbb{R}) \to T_0\mathbb{CP}^1 \) to \( 0 \in \mathbb{CP}^1 \).

5. (10 points) Let \( G \) be a Lie group acting on a manifold \( X \), and \( \mathfrak{g} \to \text{Vect}(X), v \mapsto \tilde{v} \) the corresponding infinitesimal action. Define the moment map

\[
\mu : T^*X \longrightarrow \mathfrak{g}^* \quad (\mu(x, \xi), v) = \xi(\tilde{v}_x) \quad v \in \mathfrak{g}, \xi \in T^*_x X
\]

Calculate \( \mu \) in the following cases using the identification \( T^*X \cong X \times (\mathbb{R}^n)^* \) to write the moment map in the form \( \mu(x, \xi) \), for \( x \in \mathbb{R}^n, \xi \in (\mathbb{R}^n)^* \).

(a) (2 points) Standard action \( r \cdot x = rx \) of \( G = \text{GL}(1, \mathbb{R}) \) on \( X = \mathbb{R} \).

(b) (2 points) Hyperbolic action \( r \cdot (x_1, x_2) = (rx_1, r^{-1}x_2) \) of \( G = \text{GL}(1, \mathbb{R}) \) on \( X = \mathbb{R}^2 \).

For the following cases, use the identification \( T^*G \cong G \times \mathfrak{g}^* \) induced by the identification \( TG \cong G \times \mathfrak{g} \) given by right-invariant vector fields to write the moment map in the form \( \mu(g, \xi) \), for \( g \in G, \xi \in \mathfrak{g}^* \).

(c) (2 points) Trivial action of \( G \) on itself.
(d) (2 points) Left multiplication action of $G$ on itself.

(e) (2 points) Right multiplication action of $G$ on itself.