1. (8 points) Let $G$ be a simple complex Lie group and $\rho : G \to \text{GL}(V)$ a finite-dimensional complex representation. Let $d\rho : g \to \text{End}(V)$ be the induced Lie algebra representation, and $\alpha_\rho : g \to \text{Vect}^\text{alg}(V)$ the induced infinitesimal action.

(a) (2 points) Find a natural isomorphism $\text{End}(V) \cong V \otimes V^*$.

(b) (2 points) Find a natural isomorphism $\text{Vect}^\text{alg}(V) \cong \bigoplus_{n=0}^\infty \text{Sym}^n(V^*) \otimes V$.

(c) (2 points) Explain the relation between $d\rho$ and $\alpha_\rho$ under the prior identifications.

(d) (2 points) Calculate $d\rho$ and $\alpha_\rho$ on a basis of $g = \mathfrak{sl}(2, \mathbb{C})$ for $\rho$ the standard representation of $G = \text{SL}(2, \mathbb{C})$.

2. (6 points) For each rank 2 simple Lie algebra $g$:

(a) (2 points) Draw its root system and Dynkin diagram, and explain their relationship.

(b) (2 points) Choose a Borel subalgebra $b \subset g$ and highlight the corresponding simple roots, positive roots, dominant cone, and special point $-\rho$.

(c) (2 points) Find the coefficients of each positive root as a non-negative sum of simple roots, and mark the highest weight of the adjoint representation.

3. (12 points) Let $G$ be a simple complex Lie group, and $B$ its flag variety of Borel subalgebras $b \subset g$.

(a) (2 points) Show the natural $G$-action on $B$ by conjugation induces an isomorphism $g/b \cong T_b B$ of $B$-representations for any $b \in B$.

(b) (2 points) Calculate the weight of the one-dimensional $B$-representation $\wedge^d(T_b B)$, where $d = \dim B$, in terms of the positive roots of $g$.

(c) (2 points) For any $b \in B$, show its orthogonal $b^\perp \subset g$ with respect to the Killing form is isomorphic to $n = [b, b] \subset g$ as a $B$-representation.

(d) (2 points) Set $\tilde{N} = \{(b, X) \in B \times g | X \in n = [b, b]\}$. Use the previous parts and the isomorphism $g^* \cong g$ given by the Killing form to find a $G$-equivariant isomorphism $T^* B \cong \tilde{N}$.

(e) (2 points) Show under the previous identifications the moment map $\mu : T^* B \to g^*$ of the $G$-action on $B$ is given by the projection $\tilde{N} \to g^*$, $(b, X) \mapsto X$.

(f) (2 points) For $G = \text{SL}(3, \mathbb{C})$, describe the fibers $\mu^{-1}(X)$ as concretely as possible, in particular their dimensions, for $X$ each of the matrices

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

4. (18 points) Fix $B_0 \subset \text{SL}(2, \mathbb{C})$ the Borel subgroup of upper triangular matrices, and the $\text{SL}(2, \mathbb{C})$-equivariant isomorphism $\mathbb{P}^1 \cong \text{SL}(2, \mathbb{C})/B_0$ given by acting on $\ell_0 = [1, 0] \in \mathbb{P}^1$. Consider the global sections functor

$$
\Gamma : D_{\mathbb{P}^1} \to \text{mod} \to U\mathfrak{sl}(2, \mathbb{C})
$$

from $D$-modules on the flag variety $\mathbb{P}^1 \cong \text{SL}(2, \mathbb{C})/B_0$ to $U\mathfrak{sl}(2, \mathbb{C})$-modules.
(a) (2 points) For $\ell \in \mathbb{P}^1$, consider the $D$-module $\Delta(\ell) \in D_{\mathbb{P}^1} - \text{mod}$ of delta-functions at $\ell \in \mathbb{P}^1$. Find a Borel subalgebra $\mathfrak{b} \subset \mathfrak{sl}(2, \mathbb{C})$ and a character $\chi : \mathfrak{b} \rightarrow \mathbb{C}$ and show the representation $\Gamma(\mathbb{P}^1, \Delta(\ell))$ is isomorphic to the Verma module $U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\chi$.

(b) (2 points) For $\ell \in \mathbb{P}^1$, consider the $D$-module $\Delta(U_\ell) \in D_{\mathbb{P}^1} - \text{mod}$ of algebraic distributions on $U_\ell = \mathbb{P}^1 \setminus \{\ell\}$. Find a Borel subalgebra $\mathfrak{b} \subset \mathfrak{sl}(2, \mathbb{C})$ and a character $\chi : \mathfrak{b} \rightarrow \mathbb{C}$ and show the representation $\Gamma(\mathbb{P}^1, \Delta(U_\ell))$ is isomorphic to the Verma module $U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\chi$.

(c) (6 points) Show the above constructions exhaust all $D$-modules $M \in D_{\mathbb{P}^1} - \text{mod}$ whose global sections $\Gamma(\mathbb{P}^1, M)$ are isomorphic to a Verma module.

(d) (4 points) For $\ell \in \mathbb{P}^1$, consider the $D$-module $\mathcal{O}(U_\ell) \in D_{\mathbb{P}^1} - \text{mod}$ of functions on $U_\ell = \mathbb{P}^1 \setminus \{\ell\}$. Find compatible filtrations of the $D$-module $\mathcal{O}(U_\ell)$ and the representation $\Gamma(\mathbb{P}^1, \mathcal{O}(U_\ell))$ with associated graded a sum of irreducibles.

(e) (4 points) Show that the $D$-modules $\mathcal{O}(U_\ell)$ cannot be written as a complex of $D$-modules so that under global sections the representation $\Gamma(\mathbb{P}^1, \mathcal{O}(U_\ell))$ is expressed as a complex of Verma modules.