SID (10 points): ______________ Math 250A Midterm 1 Grader SID: ______________

1. (10 points) State whether each assertion is always true (T) or sometimes false (F).
   (a) (1 point) __**T**__ For any object x of any category $\mathcal{C}$, the hom-set $\mathrm{Hom}_{\mathcal{C}}(x,x)$ is non-empty.
   (b) (1 point) __**F**__ If a functor is an equivalence, it is a bijection on objects.
   (c) (1 point) __**F**__ The Yoneda functor $Y : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$, $Y(x) = \mathrm{Hom}_{\mathcal{C}}(x, -)$ is an equivalence.
   (d) (1 point) __**F**__ All objects of a groupoid are isomorphic to each other.
   (e) (1 point) __**F**__ The forgetful functor $F : \mathrm{Grp} \to \mathrm{Set}$ is full.
   (f) (1 point) __**T**__ The forgetful functor $F : \mathrm{Grp} \to \mathrm{Set}$ is faithful.
   (g) (1 point) __**T**__ The forgetful functor $F : \mathrm{AbGrp} \to \mathrm{Grp}$ is full.
   (h) (1 point) __**T**__ The forgetful functor $F : \mathrm{AbGrp} \to \mathrm{Grp}$ is faithful.
   (i) (1 point) __**T**__ The forgetful functor $F : \mathrm{AbGrp} \to \mathrm{Grp}$ preserves products.
   (j) (1 point) __**F**__ The forgetful functor $F : \mathrm{AbGrp} \to \mathrm{Grp}$ preserves coproducts.

2. (10 points) Let $\mathcal{R}$ be the category with objects real numbers $r \in \mathbb{R}$ and hom-sets

$$\mathrm{Hom}_{\mathcal{R}}(r,s) = \begin{cases} \{\bullet\} & r \le s \\ \emptyset & r > s \end{cases}$$

   (a) (2 points) i. State (Yes or No) if the product (in the sense of category theory!) of the objects 2 and 3 exists in $\mathcal{R}$.

   i. **Yes**

   ii. If yes, calculate it.

   ii. 2

   (b) (2 points) i. State (Yes or No) if the coproduct of the objects 2 and 3 exists in $\mathcal{R}$.

   i. **Yes**

   ii. If yes, calculate it.

   ii. 2

   (c) (2 points) i. State (Yes or No) if the limit (in the sense of category theory!) of the diagram

$$\cdots \longrightarrow 1/4 \longrightarrow 1/3 \longrightarrow 1/2 \longrightarrow 1$$

   exists in $\mathcal{R}$.

   i. **Yes**

   ii. If yes, calculate it.

   ii. 0

   (d) (2 points) i. State (Yes or No) if the colimit of the diagram

$$\cdots \longrightarrow 1/4 \longrightarrow 1/3 \longrightarrow 1/2 \longrightarrow 1$$

   exists in $\mathcal{R}$.

   i. **Yes**

ii. If yes, calculate it.

ii. 1

(e) (2 points) i. State (Yes or No) if $\mathcal{R}$ is equivalent to the opposite category $\mathcal{R}^{op}$.

i. **Yes**

ii. If yes, define an equivalence $F : \mathcal{R} \to \mathcal{R}^{op}$ by specifying its map on objects.

ii. $F(r) = -r$

3. (8 points) For each diagram of abelian groups, calculate its limit.

(a) (4 points)

$$\begin{array}{ccc} & & \mathbb{Z} \\ & & \downarrow{\scriptstyle 1\times} \\ \langle 0 \rangle & \xrightarrow[0\times]{} & \mathbb{Z}/3\mathbb{Z} \end{array}$$

(a) $\mathbb{Z}$

(b) (4 points)

$$\begin{array}{ccc} & & \mathbb{Z}/3\mathbb{Z} \\ & & \downarrow{\scriptstyle 2\times} \\ \mathbb{Z}/2\mathbb{Z} & \xrightarrow[3\times]{} & \mathbb{Z}/6\mathbb{Z} \end{array}$$

(b) $\langle 0 \rangle$

4. (8 points) For each diagram of abelian groups, calculate its colimit.

(a) (4 points)

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{4\times} & \mathbb{Z} \\ {\scriptstyle 0\times}\downarrow & & \\ \langle 0 \rangle & & \end{array}$$

(a) $\mathbb{Z}/4\mathbb{Z}$

(b) (4 points)

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{1\times} & \mathbb{Z}/5\mathbb{Z} \\ {\scriptstyle 2\times}\downarrow & & \\ \mathbb{Z} & & \end{array}$$

(b) $\mathbb{Z}/10\mathbb{Z}$

5. (12 points) Let CRing be the category of commutative rings.

(a) (2 points) What is the initial object of CRing?

(a) $\mathbb{Z}$

(b) (2 points) What is the terminal object of CRing?

(b) $\langle 0 \rangle$

Recall for commutative rings R, S, their coproduct is the tensor product $R \otimes S$.

(c) (4 points) Calculate the tensor product $\mathbb{Q} \otimes \mathbb{Z}/n\mathbb{Z}$ as a function of n.

(c) $\langle 0 \rangle$

(d) (4 points) Calculate the tensor product $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$ as a function of m, n.

(d) $\mathbb{Z}/\gcd(m,n)\mathbb{Z}$

6. (12 points) For a category $\mathcal{C}$, consider the identity functor $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, i.e. the functor that takes each object $x \in \mathcal{C}$ to itself and each morphism $f : x \to y$ to itself.

   For each listed category $\mathcal{C}$, calculate the group $\mathrm{Aut}(\mathrm{Id}_{\mathcal{C}})$ of automorphisms of $\mathrm{Id}_{\mathcal{C}}$, i.e. the group of invertible natural transformations $\phi : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$.

   (a) (4 points) $\mathcal{C} = \mathrm{Vect}_k$ the category of k-vector spaces over a field k.

   (a) $k^\times$

   (b) (4 points) $\mathcal{C} = \mathrm{FinSet}$ the category of finite sets.

   (b) $\langle 1 \rangle$

   (c) (4 points) $\mathcal{C} = BH$ the classifying category of a group H, i.e. the category with one object $\bullet$ with hom-set $\mathrm{Hom}_{BH}(\bullet, \bullet) = H$ and composition given by mutiplication in H.

   (c) $Z(H)$