Math 250A Final

This is a take-home final due **by email at noon on Monday, December 10, 2018**.

- You must work independently and submit your own solutions.
- You may use any static resources: textbooks, websites, etc.

1. (6 points) Let G be a finite group.
   (a) (2 points) Show the character of the regular representation $\mathbb{C}[G]$ is the function
   $$\chi(x) = \begin{cases} |G| & x = 1 \\ 0 & x \neq 1 \end{cases}$$
   (b) (2 points) For a subgroup $H \subset G$, and representation W with character σ, show the character of the induced representation $\mathrm{Ind}_H^G(W)$ is given by
   $$\chi(x) = \frac{1}{|H|} \sum_{g \in G} \tilde{\sigma}(gxg^{-1})$$
   where $\tilde{\sigma}(y) = \sigma(y)$ when $y \in H$ and $\tilde{\sigma}(y) = 0$ when $y \notin H$.
   (c) (2 points) Explain why (a) is a special case of (b).

2. (8 points) Calculate the character table for each listed group:
   (a) (2 points) $C_8 = \langle a \mid a^8 = 1 \rangle$ the cyclic group of 8 elements.
   (b) (2 points) $D_8 = \langle \sigma, \tau \mid \sigma^2 = 1, \tau^4 = 1, \sigma\tau\sigma = \tau^{-1} \rangle$ the dihedral group of 8 elements.
   (c) (2 points) $Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = -1, ijk = -1 \rangle$ the quaternion group of 8 elements.
   (d) (2 points) For each irreducible representation V of each above group, use the character table to find the irreducible summands (with multiplicities) of $V \otimes V$.

3. (8 points) Consider the polynomial algebra $A = \mathbb{C}[x]$.
   (a) (2 points) Classify the irreducible A-modules that are finite-dimensional over $\mathbb{C}$.
   (b) (2 points) Classify the indecomposable A-modules that are finite-dimensional over $\mathbb{C}$.

   Consider the algebra $W = \mathbb{C}\langle x, y \mid xy - yx = 1 \rangle$. Thus by definition, W is the quotient of the free $\mathbb{C}$-algebra $\mathbb{C}\langle x, y \rangle$ on two generators x, y modulo the two-sided ideal generated by $xy - yx - 1$.
   (c) (2 points) Classify the irreducible W-modules that are finite-dimensional over $\mathbb{C}$.
   (d) (2 points) Classify the irreducible W-modules M on which x acts locally nilpotently: for each $m \in M$, there exists $k \geq 0$ such that $x^k m = 0$.

4. (8 points) Let A be a k-algebra, and $A - \mathrm{mod}$ its category of modules.

   Construct a natural (in the colloquial sense) k-algebra isomorphism
   $$z : Z(A) \simeq \mathrm{End}(\mathrm{Id}_{A-\mathrm{mod}})$$
   between the center of A and the endomorphisms of the identity functor of $A - \mathrm{mod}$.

5. (8 points) Let A, B be k-algebras and V an (A, B)-bimodule. Consider the functor between categories of modules
   $$\Phi_V : A - \mathrm{mod} \to B - \mathrm{mod} \quad \Phi_V(M) = V \otimes_A M$$
   (a) (2 points) Show Φ_V admits a right adjoint given by
   $$\Phi_V^r : B - \mathrm{mod} \to A - \mathrm{mod} \quad \Phi_V^r(M) = \mathrm{Hom}_B(V, M)$$

(b) (2 points) Show there is an isomorphism of functors $\Phi_V \simeq \Phi_{V'}$, for (A, B)-bimodules V, V', if and only if there is an isomorphism of bimodules $V \simeq V'$.

(c) (4 points) Suppose a functor

$$F : A - \text{mod} \to B - \text{mod}$$

preserves coproducts and surjections. Show $F \simeq \Phi_V$ for some (A, B)-bimodule V.

6. (10 points) Let G be a finite group and k an algebraically closed field (but not necessarily the complex numbers or of characteristic zero).

Let $\text{Rep}_{k,fd}(G)$ denote the category of finite-dimensional G-representations on k-vector spaces.

Let $k[G]$ be the group algebra of k-valued functions on G, equipped with convolution, and $k[G] - \text{mod}_{fd}$ its category of modules that are finite-dimensional over k.

For each of the below assertions, state whether it is always true or sometimes false. If always true, provide a proof; if sometimes false, provide a counterexample.

(a) (2 points) There is an equivalence $\text{Rep}_{k,fd}(G) \simeq k[G] - \text{mod}_{fd}$.

(b) (2 points) Every object of $\text{Rep}_{k,fd}(G)$ is a direct sum of irreducibles.

(c) (2 points) If G is not the trivial group, every irreducible object of $\text{Rep}_{k,fd}(G)$ has dimension strictly less than $|G|$.

(d) (2 points) If G is abelian, then every irreducible object of $\text{Rep}_{k,fd}(G)$ is one-dimensional.

(e) (2 points) If $G = \mathbb{Z}/n\mathbb{Z}$, then $\text{Rep}_{k,fd}(G)$ has exactly n isomorphism classes of irreducible objects.