Review Session 2 From Fourier Series to Diff Eq

Final Exam Thurs, Dec 14, 3-6 pm, RSF

Covers all material with emphasis on material covered since Midterm 2

See the posted Practice Final!

1) Other <u>inner</u> <u>prod</u>s on $\mathbb{R}^n$

$$\underline{u}, \underline{v} \rightsquigarrow \langle \underline{u}, \underline{v} \rangle$$

All are of form

$$\underline{u}, \underline{v} \rightsquigarrow \langle \underline{u}, \underline{v} \rangle_A = \underline{u}^T A \underline{v}$$

- <u>sym</u>: $A = A^T$

- pos <u>def</u>: $\langle \underline{u}, \underline{u} \rangle_A = \underline{u}^T A \underline{u} \geqslant 0$

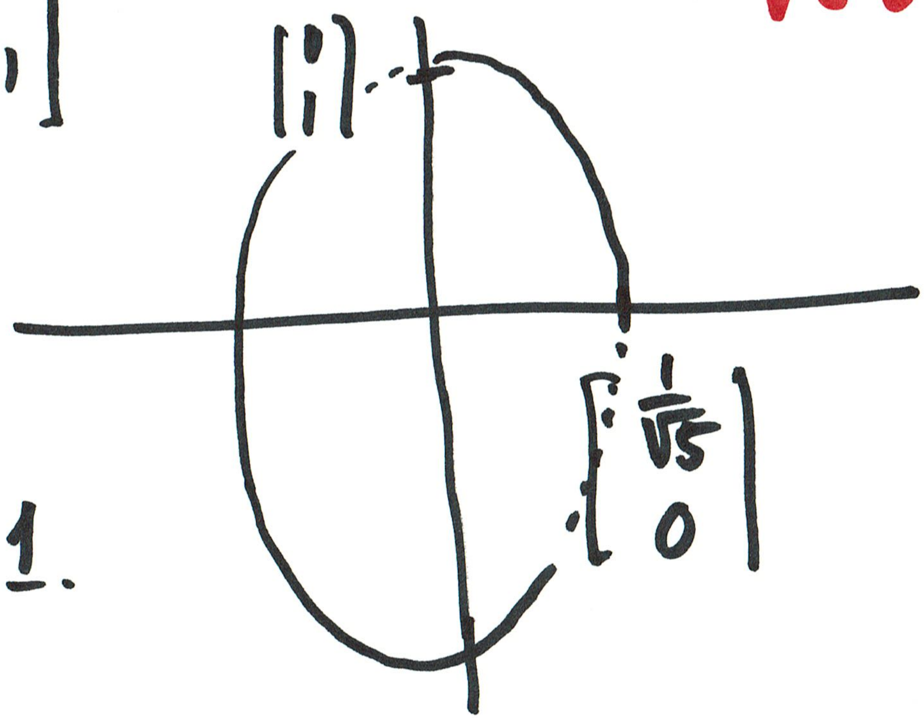
only $= 0$ when $\underline{u} = \underline{0}$

<u>Quadratic form</u> of inner prod:

$$Q_A(\underline{u}) = \langle \underline{u}, \underline{u} \rangle_A = \underline{u}^T A \underline{u}$$

"length-squared of vectors"

<u>Ex</u> $A = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$



$Q_A(\underline{u}) = \underline{1}.$

"unit ~~sphere~~ ellipse"

2) Beyond $\mathbb{R}^n$ to other vect spaces V and inner prods.

(Be sure to be able to recognize & check what are vect spaces & inner prods and what are not...)

Ex 1) $V = \mathbb{P}_5 = \{ \text{polys } p: \mathbb{R} \to \mathbb{R} \text{ of deg} \leq 5 \}$

Many possible inner prods!

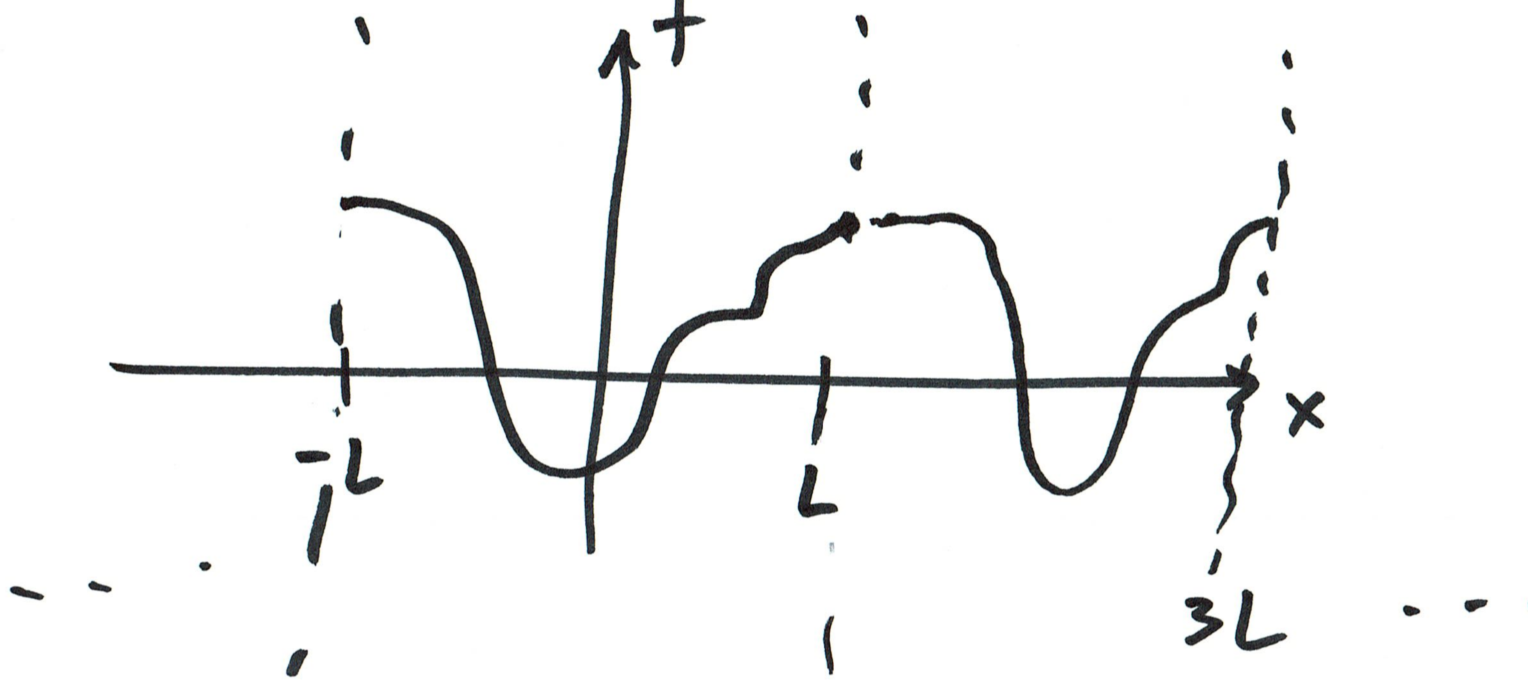
a) $\langle p, q \rangle = \int_0^1 p(x)\,q(x)\,dx$

b) $\langle p, q \rangle = \sum_{k=1}^{6=5+1} p(k)\,q(k)$

$\vdots$

Ex 2) Main example!

$V = \{ 2L\text{-periodic fns } f : \mathbb{R} \longrightarrow \mathbb{R} \}$



$= \{ \text{fns } f : [-L, L] \longrightarrow \mathbb{R} \text{ with } f(-L) = f(L) \}$

Favorite inner prod

$$\langle f, g \rangle = \int_{-L}^{L} f(x) g(x)\, dx$$

Two orthog complements

$$V^{ev}, V^{odd} \subset V$$

$$\begin{array}{cc} V^{ev} & V^{odd} \\ \ni & \ni \end{array}$$

Exs $\cos\left(\frac{n\pi x}{L}\right)$, $\sin\left(\frac{n\pi x}{L}\right)$

$n = 0, 1, 2, \ldots$ $\qquad$ $n = 1, 2, 3, \ldots$

Fourier Series informally

$\cos\left(\frac{n\pi x}{L}\right)$, $n = 0, 1, 2, \ldots$ orthog "basis" of V^{ev}

$\sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, 3, \ldots$ orthog "basis" of V^{odd}

"basis" = lin indep. + "span"

(since orthog & nonzero) (∞-lin combs)

<u>Coordinates</u> of vectors wrt these "bases"

are <u>Fourier</u> <u>coeffs</u>

$$a_n = \frac{\langle f, \cos(\frac{n\pi x}{L}) \rangle}{\langle \cos(\frac{n\pi x}{L}), \cos(\frac{n\pi x}{L}) \rangle}$$

$$= \frac{\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx}{\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)^2 dx}$$

$$= \begin{cases} \dfrac{1}{2L} \displaystyle\int_{-L}^{L} f(x) \cdot 1 \, dx & n = 0 \\ \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \cos\left(\dfrac{n\pi x}{L}\right) dx & n = 1, 2, 3, \ldots \end{cases}$$

Then $FS_f(x) = \displaystyle\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$

Alternative convention:

$$FS_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

with $$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\cdot 1 \, dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

$n = 1, 2, \ldots$

<u>Main Thm</u> f, f' p-w cont

then $$FS_f(x) = f(x)$$

if f cont at x.

What about Fourier cos, sin series on $[0,L]$?!?

Already contained in previous theory!

Given fn $f : [0, L] \to \mathbb{R}$
(ignoring value of f at $\underline{x = 0} \ldots$)
can construct $f^{ev} : [-L, L] \to \mathbb{R}$
$f^{odd} : [-L, L] \to \mathbb{R}$
with $f(x) = f^{ev}(x) = f^{odd}(x)$, $x \in [0, L]$

Check! Fourier cos series of f

$= $ Fourier series of f^{ev}

F. sin series of f

$= $ F. series of f^{odd}

<u>On to Diff</u> Eq! What kinds of eqns do we know how to solve?

1) Homog eqns (fun part!)

2) Nonhomog eqns (herein lies the pain...)

1) nth order (const coeff) ODE

2) 1st order (const coeff) systs of ODE

3) Heat eqn (PDE)

1) particular soln

2) general soln

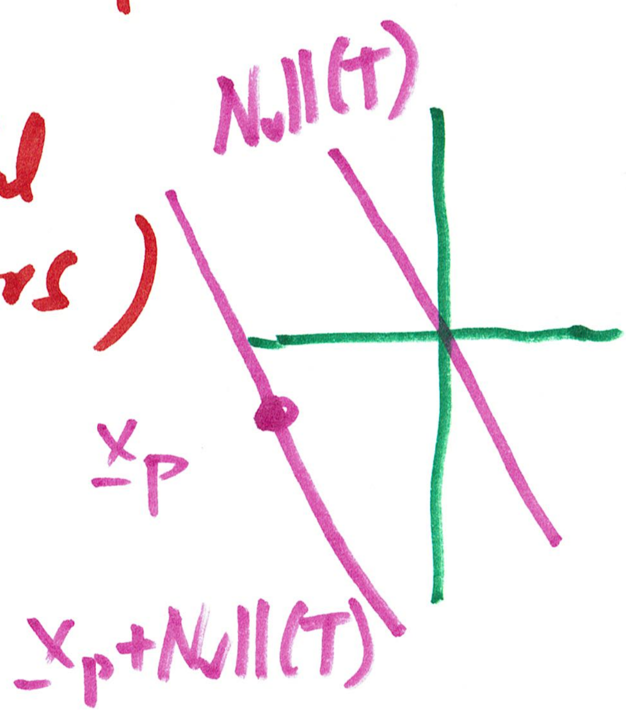
3) IVP (+BVP)

Generalities about homog vs nonhomog:

$T\underline{x} = \underline{b}$

lin transf (possibly differential operators) → T

vectors (possibly fns) → $\underline{x}$, $\underline{b}$



homog case: $\underline{b} = \underline{0}$

Soln = Null(T)

Nonhomog case: $\underline{b} \neq \underline{0}$

Soln = $\underline{x}_p$ + Null(T)

single particular soln → $\underline{x}_p$

Generalities about <u>part</u> soln, <u>gen</u> soln, <u>IVP</u> soln

1) <u>part soln</u>: single soln useful for solving nonhomog. problem.

2) <u>gen soln</u>: $\underline{x} = \underline{x}_p + c_1 \underline{v}_1 + \dots + c_k \underline{v}_k$

part soln

numbers

basis of solns to homog prob.

3) <u>IVP soln</u>: find gen soln and set $\underline{x}(0) = \underline{X}_0$ ← given data

then solve for $c_1, \dots, c_k$

Finally, the specific diff eqns:

1) n^{th} order ODE: $y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = f$

strategy: aux eqn

$$r^n + a_{n-1} r^{n-1} + \dots + a_0 = 0$$

factor and find roots $\lambda_1, \dots, \lambda_n$ (possibly repeating)

Homog version $f=0$.

distinct roots: basis of solns

$$e^{\lambda_1 t}, \ldots, e^{\lambda_n t}$$

roots with mult: k = mult of λ

$$e^{\lambda t}, te^{\lambda t}, \ldots, t^{k-1}e^{\lambda t}$$

(Euler's formula: $e^{a+ib} = e^a(\cos b + i \sin b)$)

complex exps $\longleftrightarrow$ cos, sin

Non<u>hom</u>og <u>version</u> $f \neq 0$

<u>2 methods</u>:

1) <u>Undet coeffs</u>: educated guess!

Trying to guess input fn to hit output fn.

2) <u>Var</u> of <u>params</u>: always works but must calculate $\int$ (TNSTAFL...)

2) 1^{st} order systs of ODE

$$\underline{x}'(t) = A\,\underline{x}(t) + \underline{f}(t)$$

T<u>wo</u> appr<u>oach</u>es to <u>homog version</u> $\underline{f}(t) = \underline{0}$

1) <u>Diagonalize</u> A $\rightsquigarrow$ ~~&~~ seps syst into indep eqns.

2) <u>Matrix exp</u> e^{tA} cols are basis of solns.

Non<u>homog version</u> ... analogous pair of methods.

Recall: can always transform
nth order eqn $\rightsquigarrow$ 1st order syst!

What should we have also discussed?
Wronskian and its role in
solving IVP

3) Heat eqn! $\partial_t u = \beta \partial_x^2 u$

BVP
- Dirichlet $u(0,t) = 0 = u(L,t)$
- Neumann $\partial_x u(0,t) = 0 = \partial_x u(L,t)$

IVP $u(x,0) = f(x)$

Method of soln: sep of vars.

Solns take form:

Dirichlet: $$u = \sum_{n=1}^{\infty} b_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

Neumann: $$u = \sum_{n=0}^{\infty} a_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

To solve IVP $\rightsquigarrow$ Fourier sin or cos exp of $f(x)$!