Welcome to (possibly sanity-undermining)

Lecture 8 Vector Spaces and Linear Transformations

"Mathematics is the art of giving the same name to different things."

H. Poincaré

This week

Wed Office Hrs, 12-2pm, 891 Evans

Thurs Midterm 1 Bonus Office Hrs 2-3:30pm, 740 Evans

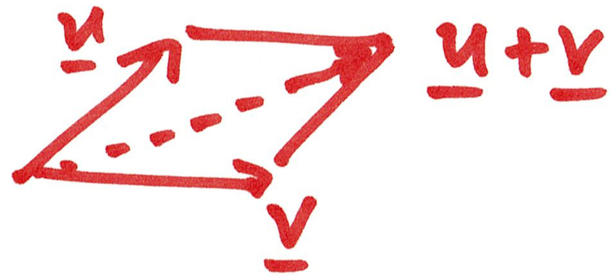
Fri Quiz through §4.2

Next week

Tue Midterm 1 during lecture meeting through §3.3

Let's list what we can do with vectors in $\mathbb{R}^n$

I) <u>Addition</u> $\underline{u}, \underline{v} \in \mathbb{R}^n \rightsquigarrow \underline{u}+\underline{v}$



1) <u>associative</u> $(\underline{u}+\underline{v})+\underline{w} = \underline{u}+(\underline{v}+\underline{w})$

2) <u>commutative</u> $\underline{u}+\underline{v} = \underline{v}+\underline{u}$

3) <u>zero vector</u> $\underline{0}$ satisfies $\underline{0}+\underline{u} = \underline{u} = \underline{u}+\underline{0}$

4) negative of vector $\underline{u} \rightsquigarrow -\underline{u}$ satisfies $\underline{u}+(-\underline{u}) = \underline{0} = (-\underline{u})+\underline{u}$

II) Scaling: $\underline{u} \in \mathbb{R}^n$, c number

$\rightsquigarrow c \cdot \underline{u}$

1) associative $\quad c \cdot (d \cdot \underline{u}) = (cd) \cdot \underline{u}$

2) unit scaling $\quad 1 \cdot \underline{u} = \underline{u}$

III) Distributivity

1) add. of scalars: $(c+d) \cdot \underline{u} = c \cdot \underline{u} + d \cdot \underline{u}$

2) add. of vectors: $c \cdot (\underline{u} + \underline{v}) = c \cdot \underline{u} + c \cdot \underline{v}$

__Def__ A __vector space__ V is a set (elements of V are called __vectors__) with two operations:

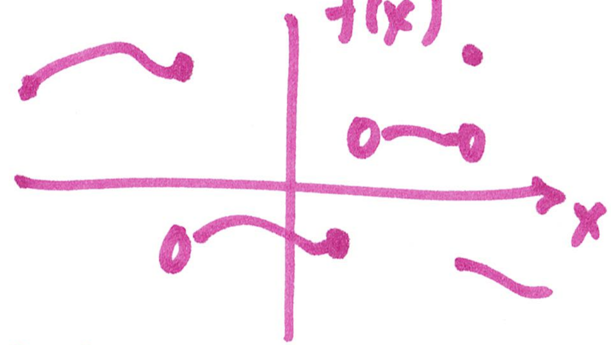
1) __addition__: $\underline{u}, \underline{v} \in V \rightsquigarrow \underline{u} + \underline{v} \in V$
2) __scaling__ $\underline{u} \in V$, c number $\rightsquigarrow c \cdot \underline{u} \in V$

__satisfying__ __the above__ __list of__ __properties__.

Examples (**Exer** : check properties hold!)

1) $V = \mathbb{R}^n$, $\underline{v} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

2) $V = \{ f : \mathbb{R} \to \mathbb{R} \}$ fns



$(f+g)(x) = f(x) + g(x)$

$(cf)(x) = cf(x)$

examples of vectors:

$f(x) = x^2 - \cos(3x)$

zero vector! $\longrightarrow$ $f(x) = 0$

3) $V = P = \{ \underset{\text{polynomial fn}}{p : \mathbb{R} \to \mathbb{R}} \}$ $p(x) = a_0 + a_1 x + \dots + a_k x^k$ some k.

examples of vectors

$p(x) = -7 + 3x^2$

$p(x) = 2x - 3x^4 + 2x^5$

zero vector! $\longrightarrow$ $p(x) = 0$

4) $V = P_n = \{ \underset{\text{poly. fn}}{p : \mathbb{R} \to \mathbb{R}} \text{ of } \deg \le n \}$

Recall deg of $p(x)$ is largest k so that $a_k \neq 0$.

by convention deg of $p(x) = 0$ is 0.

examples of vectors $n=3$

$p(x) = 2 - x + 3x^2 - 7x^3$

$p(x) = 5 - 6x^2$

zero vector! $\longrightarrow$ $p(x) = 0$

5) a) $V = S = \{ (a_1, a_2, a_3, \dots) \}$

∞ sequences of numbers

examples of vectors $(1, 0, -3, 0, 0, \pi, \sqrt{2}, \dots)$

zero vector! $\longrightarrow$ $(0, 0, 0, 0, \dots)$

b) $V = S_0 = \{ (a_1, a_2, a_3, \dots) \}$

∞ seqs of numbers <u>eventually</u> <u>zero</u>

examples of <u>vectors</u> $(7, 2, -3, 0, 2, \underline{0, 0, 0, \dots})$

<u>zero</u> <u>vector</u>! $\longrightarrow (0, 0, 0, 0, \dots)$

6) $V = M_{m \times n} = \{ m \times n \text{ matrices} \}$

examples of <u>vectors</u>: $\underline{m=2,\ n=3}$ $\quad A = \begin{bmatrix} -2 & 0 & 0 \\ 3 & 2 & 5 \end{bmatrix}$

zero vector! $\longrightarrow$ $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

⚠ 2 **Caution** Many tricky non-examples

1) $\{$ poly fns $p: \mathbb{R} \to \mathbb{R}$ of deg $= n > 0 \}$

no zero vector!

2) $\{$ $m \times n$ matrices in REF $\}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

not REF!

Key concepts for vectors in $\mathbb{R}^n$ make Sense for vectors in <u>any</u> vect sp V

1) <u>lin. combs</u>. $a_1\underline{u}_1 + \cdots + a_k\underline{u}_k$

a_i numbers

$\underline{u}_i$ vectors

2) <u>Span</u> <u>noun</u>: $\mathrm{Span}\{\underline{u}_1, \ldots, \underline{u}_k\} =$

{all lin combs}

<u>verb</u>: $\underline{u}_1, \ldots, \underline{u}_k$ spans V

if $\mathrm{Span}\{\underline{u}_1, \ldots, \underline{u}_k\} = V$

3) <u>lin dep/indep</u> $\underline{u}_1, \ldots, \underline{u}_k$ <u>lin dep</u>

if there $a_1, \ldots, a_k$ not all 0.

so that $a_1 \underline{u}_1 + \cdots + a_k \underline{u}_k = \underline{0}$

zero vector!

else <u>lin indep</u>

__Exer__ $V = P_3 = \{$ poly fns $p: \mathbb{R} \to \mathbb{R}$ of deg $\leq 3 \}$

Is vector $1+x-x^3$ in span of

$1+x^3, \; 1-2x, \; x^2-x^3$

__Soln__ Turn it into a problem we've already solved!

Need a_1, a_2, a_3 numbers so that

$$1+x-x^3 = a_1(1+x^3) + a_2(1-2x) + a_3(x^2-x^3)$$

This is 4 lin eqns in 3 unknowns!

~~[illegible]~~ const term $1 = a_1 + a_2$

lin term $1 = -2a_2$

quad term $0 = a_3$

cubic term $-1 = a_1 - a_3$

aug matrix $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & -1 \end{array}\right]$

$\rightsquigarrow$ REF has pivot in aug col so incons.
so not in span

Another key concept that makes sense for vect spaces:

A map $T : V \longrightarrow W$ is a <u>lin</u> <u>transf</u>

vect spaces!

if satisfies

1) $T(\underline{u} + \underline{v}) = T\underline{u} + T\underline{v}$ "parallelograms go to parallelograms"

2) $T(c\underline{v}) = cT(\underline{v})$ "rays go to rays"

Exer $V = P_2 = \{ \text{poly fns } p: \mathbb{R} \to \mathbb{R} \text{ of deg} \leq 2 \}$

$W = \mathbb{R}^1$

$T: V \longrightarrow W \qquad T(p(x)) = p'(-1)$

derivative evaluated at -1.

1) Is T ~~linear~~ a lin transf?

2) what are null sp and range of T?

<u>Soln</u> 1) $T(p+q) \overset{?}{=} T(p) + T(q)$

$$T(cp) \overset{?}{=} c\,T(p)$$

$$T(p+q) = (p+q)'(-1)$$

$$= (p'+q')(-1)$$

$$= p'(-1) + q'(-1) = T(p) + T(q) \checkmark$$

check 2nd requirement!

2) $\text{Null}(T) = \{ p(x) = a_0 + a_1 x + a_2 x^2$ so that $p'(-1) = 0 \}$

zero vector in $\mathbb{R}^1$!

$p'(x) = a_1 + 2a_2 x$

$p'(-1) = a_1 - 2a_2$

$\text{Null}(T) = \{ p(x) = a_0 + a_1 x + \frac{1}{2} a_1 x^2$

a_0, a_1 numbers $\}$

$\text{Image}(T) = \mathbb{R}^1$ $\quad p(x) = a_1 x \quad T(p(x)) = a_1$