Welcome to Lecture 7! How I learned to stop worrying and love determinants! 

This week: Fri Quiz through §3.1

Next week: Business as usual

Tues 9/26 Midterm 1 through §3.3

Warmup Calc det(A) for following A.

1) $A = \begin{bmatrix} -7 \end{bmatrix}$ $\quad \det(A) = -7$

2) $A = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}$ $\quad \det(A) = 2 \cdot 0 - (3)(-1)$

$= 3$

3) $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$\diagdown = +$

$\diagup = -$

Cofactor exp along row 1:

$$a \cdot \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei + bfg + cdh - (afh + bdi + ceg)$$

4) $A = \begin{bmatrix} -2 & 1 & 2 & 1 \\ 5 & 0 & -1 & 2 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$A' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 0 & -1 & 2 \\ 1 & -1 & 0 & 0 \\ -2 & 1 & 2 & 1 \end{bmatrix}$ B

$\det(A) = -\det(A')$ by row swap. (R2)

$\det(A') = 1 \cdot \det(B)$

$$B' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\det(B) = -\det(B')$$

row swap

$$\det(B') = -1 \cdot (-1-4) = 5$$

$$\rightsquigarrow \det(A) = -\det(A') = -\det(B) = \det(B') = 5$$

<u>Thm</u> A $n \times n$ matrix

We have cofactor exp for $\det(A)$ along <u>any</u> row

$$a_{k1} \cdot C_{k,1} + \cdots + a_{kn} C_{k,n}$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & & a_{nn} \end{bmatrix}$$

__Ex__ $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & -2 \end{bmatrix}$

1st __row cof exp__: $\det = 1(-2) \cancel{+} - 2\cdot 2$

$= -6$

3rd __row cof exp__: $\det = -2\cdot 1 + (-2)\cdot 2$

$= -6$

5) $A = \begin{bmatrix} \lambda_1 & * & * & * & * \\ 0 & \lambda_2 & * & * & * \\ 0 & 0 & \lambda_3 & * & * \\ 0 & 0 & 0 & \lambda_4 & * \\ 0 & 0 & 0 & 0 & \lambda_5 \end{bmatrix}$

$\det(A)$
$= \lambda_1 \cdots \lambda_5$

iterate
cof exp
along last row

<u>Thm</u> A $n \times n$ matrix, either upper Δ-ar
or lower Δ-ar

Then $\det(A) = \lambda_1 \cdots \lambda_n$ product
of diag entries

__Recall__ important behavior of det under row ops

(R1) $A \rightsquigarrow E_1 A$ $\quad E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ example

upper or lower Δ-ar

$$\det(E_1 A) = \det(A)$$

So $\det(E_1 A) = \underbrace{\det(E_1)}_{= 1} \cdot \det(A)$

(R2) $A \rightsquigarrow E_2 A$ $\quad E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ example

$$\det(E_2 A) = -\det(A)$$

So $\det(E_2 A) = \underbrace{\det(E_2)}_{=-1} \det(A)$

(R3) $A \rightsquigarrow E_3 A$ $\qquad E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ example

$\lambda \neq 0$ diagonal

$$\det(E_3 A) = \lambda \det(\overset{A}{\cancel{E_3}})$$

So $\det(E_3 A) = \underbrace{\det(E_3)}_{=\lambda} \det(A)$

Also: For A in RREF

<u>n pivots</u>: $\det(A) = 1$ since $A = I_n$

<u>$< n$ pivots</u> $\det(A) = 0$

__Thm__ Any function of $n \times n$ matrices satisfying above properties (behavior under row ops + RREF calculation) is equal to det!

__Why?__ $A = E_k \cdots E_2 E_1 A_{rref}$

$$\det(A) = \det(E_k) \cdot \det(E_{k-1} \cdots E_1 A_{rref})$$

$$\cdots = \det(E_k) \cdots \det(E_1) \det(A_{rref})$$

$\blacksquare$

Thm A, B $n \times n$ matrices

$$\det(AB) = \det(A)\det(B)$$

Pf. $A = E_k \cdots E_1 A_{rref}$

Cases — A_{rref} $< n$ pivots so $\det(A) = 0$

But also $\det(AB) = 0$ since

for instance A not surj

$\Rightarrow$ AB not surj.

$\underline{A_{rref} = I_n \quad n \text{ pivots}}$

$$\det(E_k \cdots E_1 I_n)\det(B) \stackrel{?}{=} \det(E_k \cdots E_1 I_n B)$$

$$\det(E_k \cdots E_1 I_n B) = \det(E_k \cdots E_1 B)$$

$$= \det(E_k)\det(E_{k-1} \cdots E_1 B) \quad \cdots$$

$$= \det(E_k)\det(E_{k-1}) \cdots \det(E_1)\det(B)$$

Also have similarly

$$\det(E_k \cdots E_1 I_n) = \det(E_k)\det(E_{k-1} \cdots E_1)$$

$$= \cdots = \det(E_k) \cdots \det(E_1)$$

We're done!

Thm $\det(A^T) = \det(A)$

for A $n \times n$ matrix

$A = (a_{ij}) \quad A^T = (a_{ji})$

Pf $A = E_k \cdots E_1 A_{rref}$

$\det(A) = \det(E_k) \cdots \det(E_1) \det(A_{rref})$

$A^T = A_{rref}^T E_1^T \cdots E_k^T$

$\det(A^T) = \det(A_{rref}^T) \det(E_1^T) \cdots \det(E_k^T)$

Only <u>remains to check:</u>

- $\det(A_{rref}^T) = \det(A_{rref})$

- $\det(E_i^T) = \det(E_i)$

Do it once in your life!

<u>Corollary</u> We also can calc det by cofactor expansion along any column!