Welcome to Lecture 3 Linear Combinations, Spans, Linear Independence!

This week: Fri: Quiz through §1.4

Next week: HW due M, W

Fri: Quiz through §1.9

"One of the reasons there are so many terms for conditions of ice is that the mariners observing it were often trapped in it, and had nothing to do except look at it." Alec Wilkinson

Notation First examples of <u>vector spaces</u>

$\mathbb{R}^m$ = set of m-vectors with real components

$$\underline{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \qquad a_i \in \mathbb{R}$$

$\mathbb{C}^m$ = - - - - - -

- - - - - complex components

$$\underline{u} = \begin{bmatrix} a_1 + ib_1 \\ \vdots \\ a_m + ib_m \end{bmatrix} \qquad a_i, b_i \in \mathbb{R}$$

Basic operations for m-vectors:

1) Add $\underline{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$, $\underline{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ $\rightsquigarrow \underline{u} + \underline{v} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_m + b_m \end{bmatrix}$

2) Scale by number $c \cdot \underline{u} = \begin{bmatrix} c a_1 \\ \vdots \\ c a_m \end{bmatrix}$, c number

3) Dot product $\underline{u} \cdot \underline{v} = a_1 b_1 + \dots + a_m b_m$

Satisfy many helpful properties

New perspective on lin systs: vector eqns

$$m\left\{\underbrace{\left[\begin{array}{cccc|c} | & | & & | & \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n & \underline{b} \\ | & | & & | & \end{array}\right]}_{n+1}\right. \qquad \underline{v}_i, \underline{b} \in \mathbb{R}^m$$

Lin syst is same as vector eqn

$$x_1 \underline{v}_1 + x_2 \underline{v}_2 + \cdots + x_n \underline{v}_n = \underline{b}$$

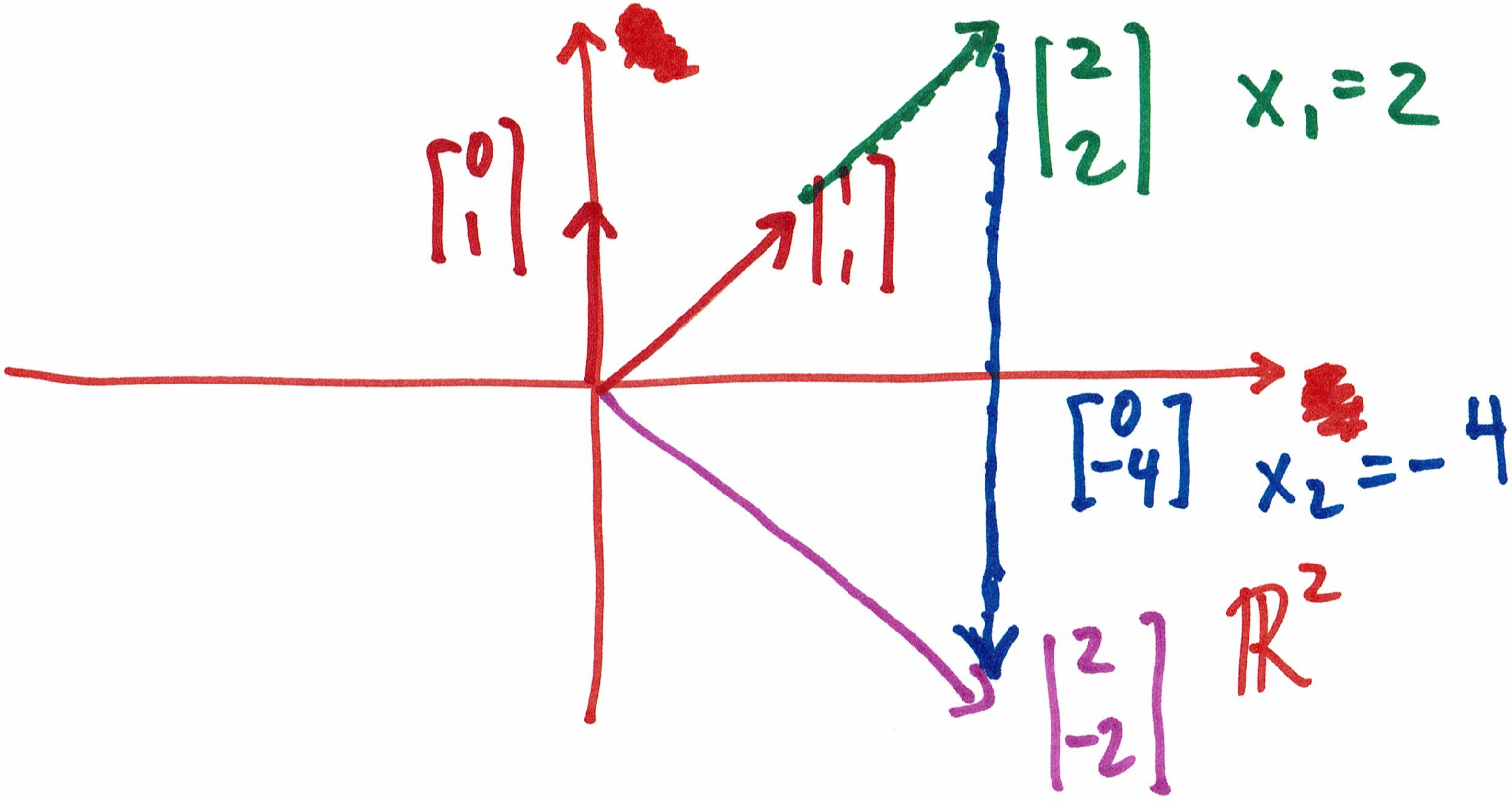
scalar variables

Exer Solve vect eqn

$$x_1\begin{bmatrix}1\\1\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\-2\end{bmatrix}$$

$m=2$

(as lin syst: $\left[\begin{array}{cc|c}1 & 0 & 2\\1 & 1 & -2\end{array}\right]$)



Def $\underline{u}$ is a **linear combination** of $\underline{v}_1, \ldots, \underline{v}_k$ if there are $a_1, \ldots, a_k$ so that

$$\underline{u} = a_1 \underline{v}_1 + \cdots + a_k \underline{v}_k$$

Observe $\underline{u}$ is a lin comb of $\underline{v}_1, \ldots, \underline{v}_k$

$$\Updownarrow$$

$$\left[\begin{array}{cccc|c} | & | & & | & \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_k & \underline{u} \\ | & | & & | & \end{array}\right]$$

is consistent (has a soln)

Exer For what c is $\underline{u}$ a lin comb of $\underline{v}_1, \underline{v}_2$?

$$\underline{u} = \begin{bmatrix} 1 \\ 1 \\ c \end{bmatrix} \qquad \underline{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Soln For what c is lin syst consistent

$$\left[\begin{array}{cc:c} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & c \end{array}\right] \quad ?$$

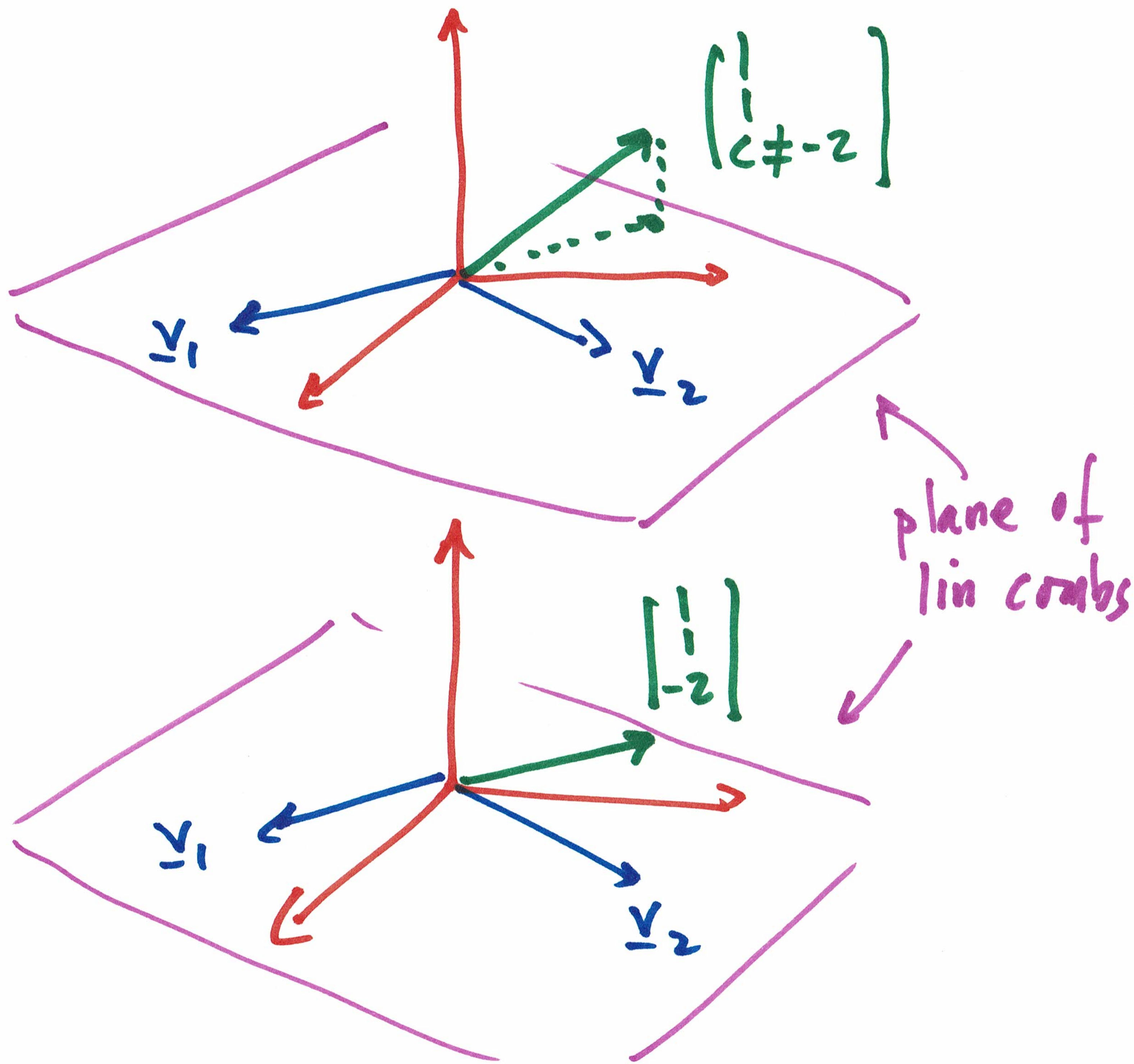
Cases

- $c \neq -2$ pivot in last col of REF so no soln
- $c = -2$ soln $x_1 = 1$, $x_2 = 2$

Picture

$c \neq -2$

$c = -2$



__Def__ (noun) **span** of $\underline{v}_1, \dots, \underline{v}_k$ is

$$\text{Span}\{\underline{v}_1, \dots, \underline{v}_k\} = \left\{ \begin{array}{c} \text{all lin combs of} \\ \underline{v}_1, \dots, \underline{v}_k \end{array} \right\}$$

$$= \left\{ \begin{array}{c} a_1\underline{v}_1 + \dots + a_k\underline{v}_k \\ a_i \text{ numbers} \end{array} \right\}$$

$$\subseteq \mathbb{R}^m$$

m = size of vectors.

" is a subset of "

<u>Observe</u> $\underline{u}$ is in $\text{Span}\{\underline{v}_1, \ldots, \underline{v}_k\}$

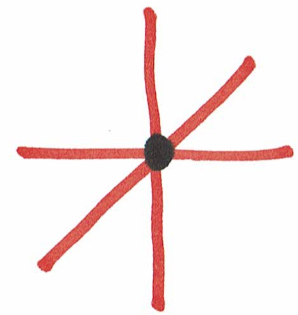
$$\Updownarrow$$

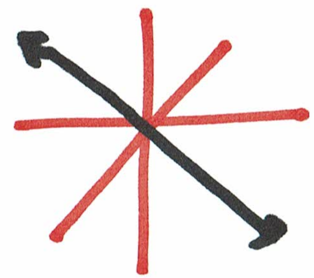
$$\left[\begin{array}{cccc|c} | & | & & | & \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_k & \underline{u} \\ | & | & & | & \end{array}\right] \text{ is consistent}$$

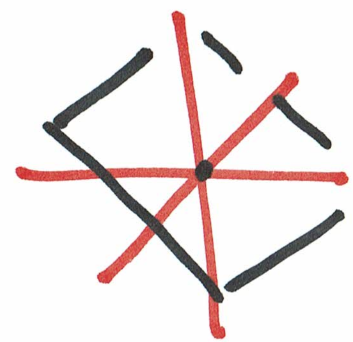
What kind of subset of $\mathbb{R}^n$ is a span?

(... to be made more precise ...)

Possibilities:

1) $\{\underline{0}\}$ origin alone 

2) line through origin 

3) plane through origin 

$\vdots$

Def (verb) $\underline{v}_1, \dots, \underline{v}_k$ sp<u>an</u>s $\mathbb{R}^m$

if $\text{Span}\{\underline{v}_1, \dots, \underline{v}_k\} = \mathbb{R}^m$

Every vector in $\mathbb{R}^m$ is a lin comb of $\underline{v}_1, \dots, \underline{v}_k$

<u>Observe</u> $\overset{\text{m-vectors}}{\underline{v}_1, \dots, \underline{v}_k}$ spans $\mathbb{R}^m$

$$\Updownarrow$$

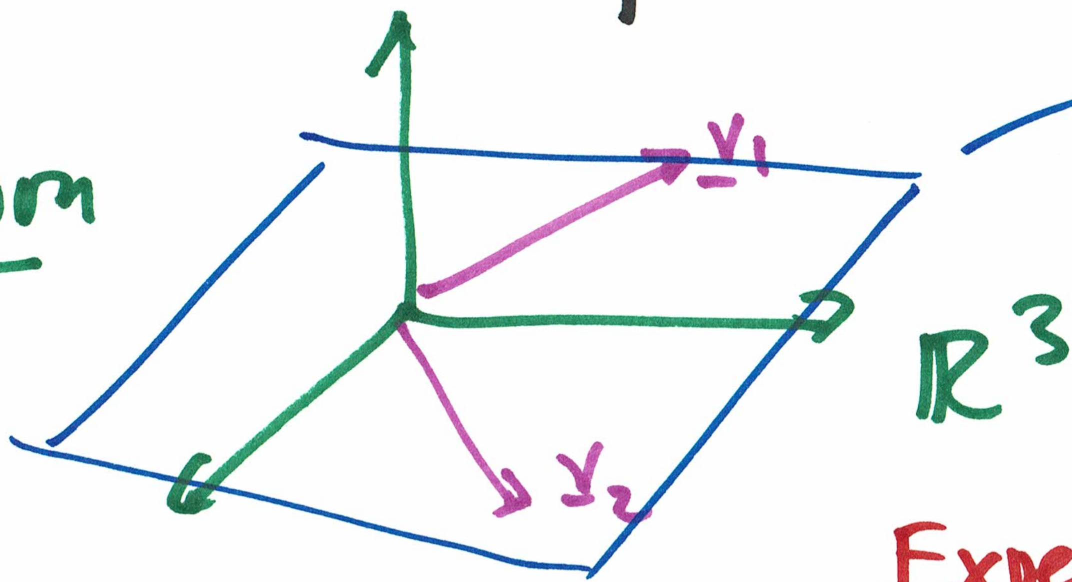
$$\left[\begin{array}{ccc|c} | & & | & \\ \underline{v}_1 & \cdots & \underline{v}_k & \underline{u} \\ | & & | & \end{array}\right]$$ is consistent for any $\underline{u} \in \mathbb{R}^m$

$$\Updownarrow$$

coeff matrix $\begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_k \\ | & & | \end{bmatrix}$ has a pivot in each row so m pivots

__Exer__ Is it possible for two vectors $\underline{v}_1, \underline{v}_2$ to span $\mathbb{R}^3$?

__Cartoon__



__Expect__: biggest possible Span is plane

__Expect__ $\text{Span}\{\underline{v}_1, \underline{v}_2\} \neq \mathbb{R}^3$

Let's convince ourselves $\operatorname{Span}\{\underline{v}_1, \underline{v}_2\} \neq \mathbb{R}^3$

Can we solve $\left[\begin{array}{cc|c} | & | & | \\ \underline{v}_1 & \underline{v}_2 & \underline{b} \\ | & | & | \end{array}\right]$ for any $\underline{b} \in \mathbb{R}^3$?

$\xrightarrow{\text{row reduce}}$ RREF $\left[\begin{array}{cc|c} | & | & | \\ \underline{w}_1 & \underline{w}_2 & \underline{y} \\ | & | & | \end{array}\right]$

Possibilities for coeff matrix: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Never is there a pivot in each row!

(3 rows, but 2 cols so at most 2 pivots)

Take $\underline{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ then lin syst is inconsistent!

Find $\underline{b}$ by inverting row ops starting from $\underline{y}$.

Main conclusion If $\underline{v}_1, \ldots, \underline{v}_k$ spans $\mathbb{R}^m$ then must have $k \geq m$!

Caution If $k \geq m$, not nec true that $\underline{v}_1, \ldots, \underline{v}_k$ spans $\mathbb{R}^m$

Ex $\mathbb{R}^3$, $\underline{v}_1 = \underline{v}_2 = \ldots = \underline{v}_{10^6} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$m = 3$, $k = 10^6$

__Def__ 1) $\underline{v}_1, \dots, \underline{v}_k$ is __linearly dependent__

if there are numbers $a_1, \dots, a_k$

not all 0 (at least one a_i is not zero)

such that $a_1 \underline{v}_1 + \dots + a_k \underline{v}_k = \underline{0}$

2) Else $\underline{v}_1, \dots, \underline{v}_k$ are linearly __in__dependent

What does lin indep mean directly?

$\underline{v}_1, \dots, \underline{v}_k$ is lin indep means

whenever $a_1 \underline{v}_1 + \dots + a_k \underline{v}_k = \underline{0}$

we have $a_1 = \dots = a_k = 0$

__Observe__ $\underline{v}_1, \dots, \underline{v}_k$ are lin dep

$$\Updownarrow$$

$$\left[\begin{array}{ccc|c} | & & | & \\ \underline{v}_1 & \cdots & \underline{v}_k & \underline{0} \\ | & & | & \end{array}\right]$$ has soln $(a_1, \dots, a_k)$ some $a_i \neq 0$.

(Always have soln $(0, \dots, 0)$)

Summary for list $\underline{v}_1, \dots, \underline{v}_k \in \mathbb{R}^m$

Spans $\mathbb{R}^m$?	Lin Indep?
Want many vectors in small space Need $k \geq m$	Want few vectors in big space Need $k \leq m$
Adding vectors to list only helps	Deleting vectors from list only helps
$\left[\begin{array}{ccc:c} \mid & & \mid & \\ \underline{v}_1 & \cdots & \underline{v}_k & \underline{b} \\ \mid & & \mid & \end{array}\right]$ has soln any $\underline{b}$ pivot in each row of coeffs	$\left[\begin{array}{ccc:c} \mid & & \mid & \\ \underline{v}_1 & \cdots & \underline{v}_k & \underline{0} \\ \mid & & \mid & \end{array}\right]$ has unique soln $\underline{0}$ pivot in each col of coeffs