Look at it. Alec Wilkinson in it, and had nothing to do except
mariners observing if we are often trapped
terms for conditions of ice is that the
One of the reasons there are so many

Fru: Give through 8.1.9

Next week: How the M, W

This week: T?: Give through 8.1.4

Spans, Linear Independence,

Welcome to Lecture 3  Linear Combinations,
Notation

$\mathbb{R}^m = \text{set of } m\text{-vectors with real components}$

$u = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \quad a_i \in \mathbb{R}$

$\mathbb{C}^m = \text{complex components}$

$u = \begin{bmatrix} a_i + ib_1 \\ \vdots \\ a_m + ib_m \end{bmatrix} \quad a_i, b_i \in \mathbb{R}$
Satisfy many helpful properties

3) Dot product $\overline{u} \cdot \overline{v} = a_1 b_1 + \ldots + a_n b_n$

2) Scalar by number $c \overline{u} = [c a_1, c a_2, \ldots, c a_m]$

1) All $\overline{u}$ vectors for $m$ vectors:

Basic operations for $m$ vectors:
Scalar variables:

\[ x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = b \]

Lin. syst. is same as vector eqn.

\[ \forall \mathbf{v}, \mathbf{b} \in \mathbb{R}^m \]

New perspective on Lin. syst.: vector ans.
Def \( \mathbf{u} \) is a linear combination of \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) if there are \( a_1, a_2, \ldots, a_k \) such that

\[
\begin{bmatrix}
\mathbf{u}_1 \\
\vdots \\
\mathbf{u}_n
\end{bmatrix} = a_1 \begin{bmatrix}
\mathbf{v}_1 \\
\vdots \\
\mathbf{v}_1
\end{bmatrix} + \cdots + a_k \begin{bmatrix}
\mathbf{v}_k \\
\vdots \\
\mathbf{v}_k
\end{bmatrix}
\]

It is consistent (has a soln) if \( \mathbf{A} \mathbf{x} = \mathbf{b} \) has a soln for some \( \mathbf{b} \). Observe \( \mathbf{u} \) is a lin comb of \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) in \( \mathbf{v} \).
\[ c = 2 \] so no soln in last col of REF.

\[ \begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \]

Ex: For what c is Lin consis?

\[ \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ \lambda \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} \lambda \\ -1 \\ c \end{bmatrix} \]
$c = -2$

Diagram with arrows and labels indicating relationships and equations.
"" is a subset of \( \mathbb{R}^m \) = span of \( \{ \vec{v}_1, \ldots, \vec{v}_n \} \).

\[ \text{Def (noun)} \text{ span of } \vec{v}_1, \ldots, \vec{v}_n \]
Inconsistent

\[
\begin{pmatrix}
\overline{\eta} & u_1 & \cdots & u_n & \cdots \\hline
1 & \alpha_2 & \cdots & \alpha_n & \cdots
\end{pmatrix}
\]

Observe \( \overline{\eta} \) is in \( \text{Span} u_1, \ldots, u_n \) \( \forall \epsilon \)
3) Plane through origin
2) Line through origin
1) \( \varnothing \) origin alone

Possible choices:

... to be made more precise ...

What kind of subset of \( \mathbb{R}^n \) is a span?
Every vector in \( \mathbb{R}^n \) in a lin comp if \( \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_k \} = \mathbb{R}^n \)
a pivot in each row so w.m. pivots

\[
\begin{pmatrix}
1 & 1 & 1 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\end{pmatrix}
\]

has

corresponding matrix is consistent for any \( x \in \mathbb{R}^m \)

Observe \( \frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \ldots, \frac{1}{\sqrt{m}} \) spans \( \mathbb{R}^m \)
Exer: Is it possible for two vectors $v_1, v_2$ to span $\mathbb{R}^3$?

- Cannot span $\mathbb{R}^3$.

- Expect: biggest possible space is plane.

- Expect $\text{Span}(v_1, v_2) \neq \mathbb{R}^3$. 
Let's convince ourselves span $R^3$.

Can we solve \[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 2 \\
1 & 1 & 3
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}
\]
for any $R^3$?

Possibilities for core matrix:
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
reduce row RREF
\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
Never is there a pivot in each row!

(3 rows, but 2 cols so at most 2 pivots)

Take $y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ then lin syst is inconsistent!

Find $b$ by inverting row ops from starting from $y$. 
Ex \( R^3 \), \( y_1 = y_2 = \ldots = y_{106} = 0 \). 

Main conclusion: If \( k \geq m \), not nec true.

Then must have \( k \geq m \).
Def (1) \( \mathbf{y}_1, \ldots, \mathbf{y}_k \) is linearly dependent

\[ \text{if there are numbers } a_1, \ldots, a_k \text{ not all zero} \]

\[ a_1 \mathbf{y}_1 + \ldots + a_k \mathbf{y}_k = \mathbf{0} \]

Else \( \mathbf{y}_1, \ldots, \mathbf{y}_k \) are linearly independent

2) Else \( \mathbf{y}_1, \ldots, \mathbf{y}_k \) are linearly independent
we have $q_1 = \ldots = q_k = 0$ whenever $a_1 \lor \ldots \lor a_k = 0$.

What does $\lor$ in $\text{lin indep}$ mean?
Always have soln \((0, \ldots, 0)\).

Some \(a_i \neq 0\).


\[
\begin{bmatrix}
0 \\
\frac{1}{c} \\
\vdots \\
\frac{1}{b}
\end{bmatrix}
\]

Has soln \([a_1, \ldots, a_k] \neq 0\).
<table>
<thead>
<tr>
<th>Subtracting vectors to ( \text{dim} ) ( \text{basis} )</th>
<th>Adding vectors to ( \text{dim} ) ( \text{basis} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Need } k \in \mathbb{N} ) ( \text{in } \mathbb{R}^n )</td>
<td>( \text{Need } k \in \mathbb{N} ) ( \text{in } \mathbb{R}^n )</td>
</tr>
<tr>
<td>( \text{Want few vectors} )</td>
<td>( \text{Want many vectors} )</td>
</tr>
<tr>
<td>( \text{Lin Indep?} )</td>
<td>( \text{Spans } \mathbb{R}^n ? )</td>
</tr>
<tr>
<td>( \sum \text{for first } v_1, \ldots, v_k \in \mathbb{R}^m )</td>
<td></td>
</tr>
</tbody>
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