Lecture 26

Parting is such sweet sorrow... but we'll always have the Heat Eqn!

Fri Last Quiz!
through §10.4

Next week Review during lecture meetings
Wed Off Hrs 12-2pm, 891 Evans

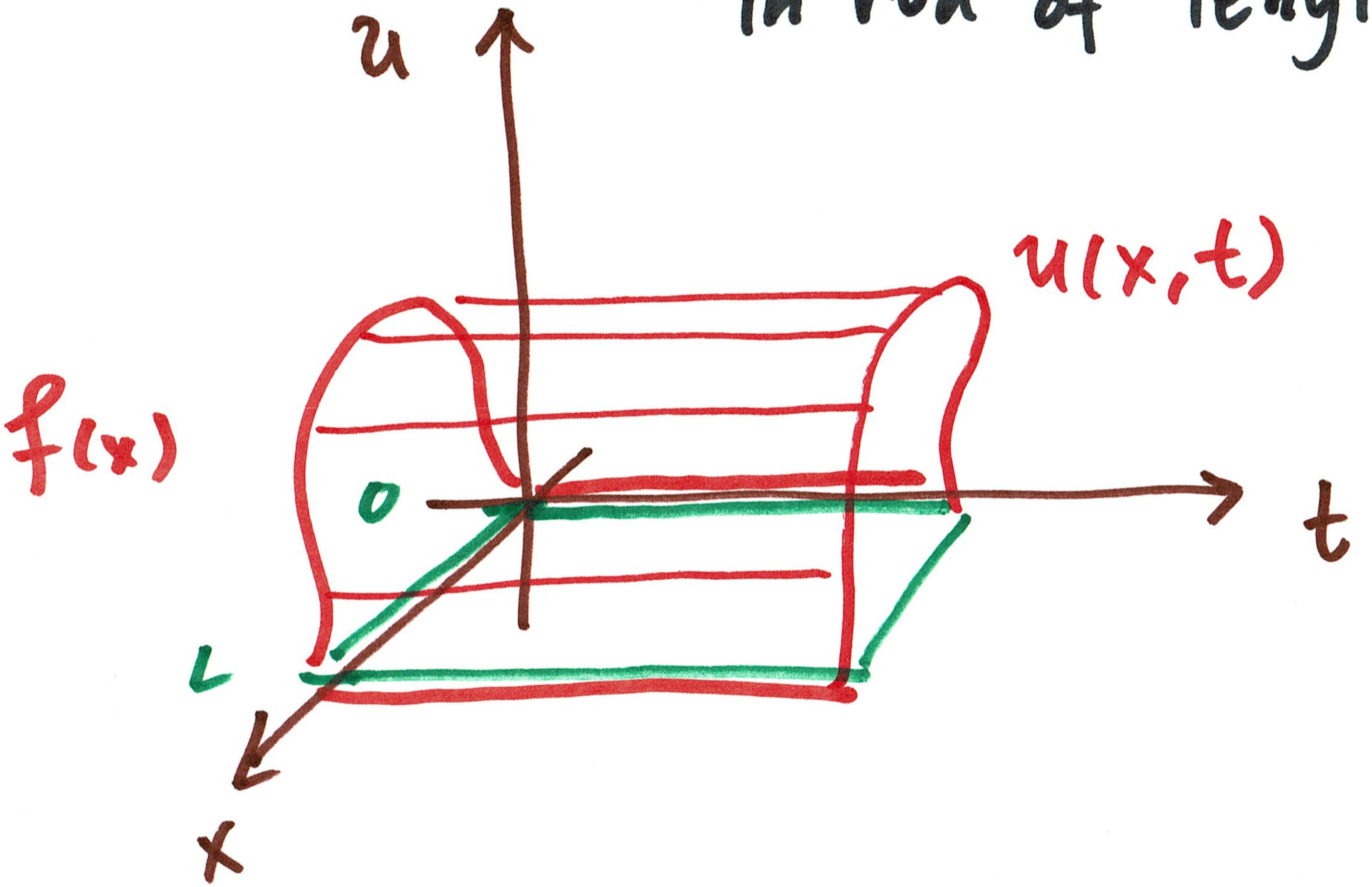
Final: Thurs, Dec 14, 3-6 pm, RSF

Good luck!!!

"It's not the heat; it's the humility."
— Yogi Berra

Heat Eqn

temp $u(\underset{\substack{\nearrow \\ \text{position} \\ \text{in rod of length } L}}{x}, \underset{\nwarrow \text{ time}}{t})$



Equations $\partial_t u = \beta \, \partial_x^2 u$

$\uparrow$ const depending on material of rod

IVP: $u(x,0) = f(x)$ $\leftarrow$ init. temp.

BVP - 2 versions:

1) Dirichlet: $u(0,t) = u(L,t) = 0$ $\swarrow$ const temp at ends of rod

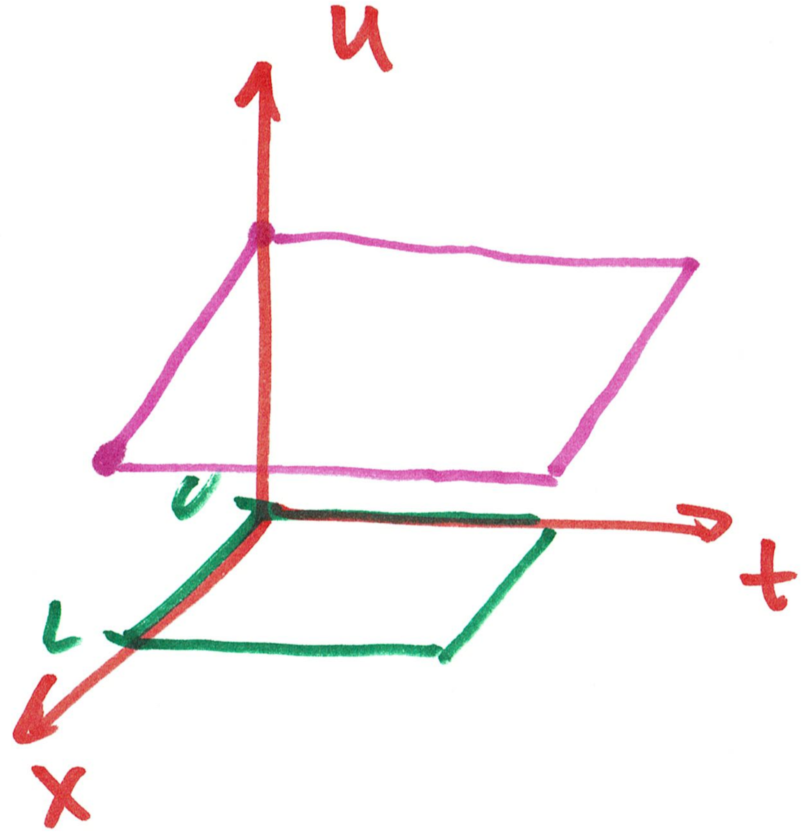
2) Neumann: $\partial_x u(0,t) = \partial_x u(L,t) = 0$

$\nwarrow$ rod ends are insulated

Heat eqn $\partial_t u = \beta \partial_x^2 u$ always has steady state solns:

indep of t!

$$u(x,t) = ax + b$$



Method Separation of variables

Dream of soln of form:

$$u(x,t) = X(x)\,T(t)$$

Why? Reduce partial d.eq ($\partial_t \& \partial_x$)
to 2 separate ordinary d.eqs
($\partial_t \; \cdots \; \partial_x$)

Plug $u(x,t) = X(x)T(t)$ into heat eqn

$$\partial_t u = X(x)T'(t)$$

$$\partial_x^2 u = X''(x)T(t)$$

Heat eqn: $X(x)T'(t) = \beta X''(x)T(t)$

$$\frac{1}{\beta}\underset{\substack{\uparrow \\ \text{fn of t only!}}}{\frac{T'(t)}{T(t)}} = \underset{\substack{\uparrow \\ \text{fn of x only!}}}{\frac{X''(x)}{X(x)}} \quad \overset{\text{both const!}}{\Nearrow}$$

+

Let $-\lambda$ be the const so that

$$\frac{1}{\beta}\frac{T'(t)}{T(t)} = -\lambda = \frac{X''(x)}{X(x)}$$

Decoupled into separate ordin diff eqs!

$$(*) \quad T'(t) + \lambda\beta T(t) = 0 \qquad \underline{\underline{\text{1st order}}}$$

$$(**) \quad X''(x) + \lambda X(x) = 0 \qquad \underline{\underline{\text{2nd order}}}$$

<u>Solving</u> (*) $T(t) = ce^{-\beta\lambda t}$ general soln.

<u>Solving</u> (**) <u>aux</u> eqn $r^2 + \lambda = 0$

factor $(r - i\sqrt{\lambda})(r + i\sqrt{\lambda}) = 0$

$X(x) = c_1 e^{i\sqrt{\lambda}\, x} + c_2 e^{-i\sqrt{\lambda}\, x}$ general soln.

Altogether $u(x,t) = X(x)\, T(t)$
solves heat eqn! <u>for any</u> $\underline{\lambda}$

To choose a soln, we must consider $\frac{\underline{\text{BVP}}}{\underline{\text{IVP}}}$

<u>Dirichlet BVP</u>

$$\begin{array}{ccccc} u(0,t) & = & 0 & = & u(L,t) \\ \parallel & & & & \parallel \\ X(0)T(t) & & & & X(L)T(t) \end{array}$$

<u>Either</u> 1) $T(t) = 0$ so $u(x,t) = 0$

<u>or</u> 2) $X(0) = 0 = X(L)$

Recall $X(x) = c_1 e^{i\sqrt{\lambda} x} + c_2 e^{-i\sqrt{\lambda} x}$

$$X(0) = c_1 + c_2 = 0$$

$$X(L) = c_1 e^{i\sqrt{\lambda} L} + c_2 e^{-i\sqrt{\lambda} L} = 0$$

<u>Lin syst</u> for c_1, c_2!

$$\left[\begin{array}{cc:c} 1 & 1 & 0 \\ e^{i\sqrt{\lambda} L} & e^{-i\sqrt{\lambda} L} & 0 \end{array}\right]$$

Possibilities:

1) $\lambda = 0$: $X(x) = c_1 + c_2$, $\partial_x^2 X(x) = 0$

so $T(t)$ const.

(steady state...)

2) $\lambda < 0$: Exer : lin syst has single soln $c_1 = c_2 = 0$

so $X(x) = 0$ so $u(x,t) = 0$

Possibilities 1) & 2) are not interesting!

3) **$\lambda > 0$** can rewrite $X(x)$ in form

$$X(x) = d_1 \cos(\sqrt{\lambda}\, x) + d_2 \sin(\sqrt{\lambda}\, x)$$

Dirichlet BVP: $X(0) = d_1 = 0$

so $X(L) = d_2 \sin(\sqrt{\lambda}\, L) = 0$

Either: $d_2 = 0$ so $X(x) = 0$ so $u(x,t) = 0$ uninteresting

or: $\sqrt{\lambda} = \frac{n\pi}{L}$!

__Conclusion__:

__basis of solns__ for $X(x)$:

$$\sin\left(\frac{n\pi}{L}x\right), \quad n=1,2,3,\ldots$$

__basis of solns__ for heat eqn & Dirichlet BVP

$$e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right), \quad n=1,2,3,\ldots$$

Similarly for Neumann BVP

basis of solns for X(x) :

$$\cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, \dots$$

basis of solns for heat eqn & Neumann BVP

$$e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, \dots$$

All <u>that remains</u>: IVP $u(x,0) = f(x)$

$u(x,0)$ is in "span" of $\sin\left(\frac{n\pi x}{L}\right)$,

for Dirichlet BVP $n = 1, 2, 3, \ldots$

If we can write $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

then $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$

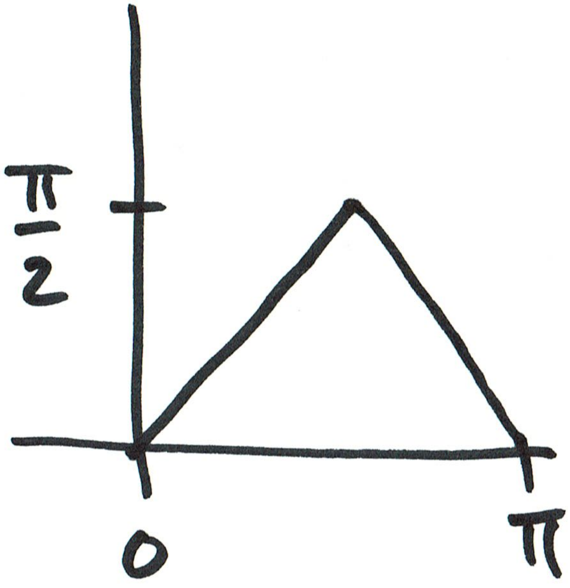
will solve IVP!

Exer Solve heat eqn on rod of length $L=\pi$

Dirichlet BVP : $u(0,t) = 0 = u(L,t)$

IVP : $u(x,t) =$

$$\begin{cases} x & x \leq \frac{\pi}{2} \\ \pi - x & x > \frac{\pi}{2} \end{cases}$$



Soln $$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

It remains to determine b_n.

How? Fourier series!

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Fourier sin series of $f(x)$!

Need $f(x)$

Calculate . . .

$$b_n = \begin{cases} 0 & n \text{ even} \\ \frac{4}{\pi n^2}(-1)^{(n-1)/2} & n \text{ odd} \end{cases}$$

Final answer :

$$u(x,t) = \frac{4}{\pi}\left(e^{-t}\sin(x) - \frac{1}{9}e^{-9t}\sin(3x) + \frac{1}{25}e^{-25t}\sin(5x) - \dots\right)$$