"It's not the heat, it's the humidity."

Good luck!!!

Final: Thurs, Dec 14, 3-6pm, RSE

Next week: Review during lecture meetings through 12/9.

End last quiz!

"The Heat is on!"

SORROW... but we'll always have Lecture 26 Painting is such sweet...
Heat Eqn Temp u(x,t) in rod of length L at position x and time t.
Equations $\partial^2 u / \partial t^2 = \alpha \partial^2 u / \partial x^2$

BVP - 2 versions:
1) Dirichlet: $u(x,0) = f(x)$
2) Neumann: $\partial u / \partial t(x,0) = g(x)$

IVP:
$u(x,0) = 0$
$u(0,t) = 0$
$\partial u / \partial x(0,t) = 0$
$x = L$
\partial u / \partial x(L,t) = 0$
\partial u / \partial x(0,t) = 0$

Insulated on both ends.

Not insulated on both ends.

init. temp.

const temp.

From material properties:
const depending on material

Formulas for rod of

$u(x,t) = \sum_{n=0}^{\infty} a_n \sin \left( \frac{n \pi x}{L} \right) e^{-\alpha n^2 \pi^2 t / L^2}$

$u(x,0) = f(x)$

Heat eqn \( \frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \) always has

steady state solns:

\( u(x,t) = e^{-\frac{x^2}{\alpha t}} \)
$$(x^2 + e^x \ldots e^x)$$

To separate ordinary des.

Why? Reduce partial des. (separable)

$$(x^2 + e^x) \frac{\text{d}x}{\text{d}t} = x(x) \frac{\text{d}t}{\text{d}t}$$

Method of soln of form:

D.V.. Separation of variables
Plug \( u(x,t) = X(x)T(t) \) into heat eqn

\[ \partial_t u = X(x)T'(t) \]
\[ \partial_x u = X''(x)T(t) \]

Heat eqn: \( X(x)T'(t) = \beta X''(x)T(t) \)

\[ \frac{1}{T'(t)} \cdot \frac{T'(t)}{\beta} = \frac{X''(x)}{X(x)} \]

both const!

fn of t only! + fn of x only!
\[ T'(t) + \gamma PT(t) = 0 \]

Decoupled into separate 2nd order 1st order diff eqs.

Let \( T \) be the constant so that

\[ \frac{1}{\Gamma(T)} = \gamma \]

\[ \frac{1}{\Gamma(T)} = -\gamma \]

\[ \frac{1}{T'}(T) = X(x) \]
To choose a solution, we must consider
\[ \frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0 \]

Solves heat eqn./ for any \( \theta \)
\[ u(x,t) = X(x)T(t) \]

Altogether \( u(x,t) \) general \( X(x) = c'e^{i\alpha x} + c''e^{-i\beta x} \)

Factor \( (\alpha - i\beta)(\alpha + i\beta) = 0 \)

Solving \( \alpha = 0 \) aux eqn. \( r^2 + \xi = 0 \)

Solving \( \xi = \) general

General solution \( V(r,\theta, z, t) = ce^{-\alpha^2 t} \)
\[ x(0) = 0 = x(l) \]
Lin. syst. for $c_1, c_2$

\[
\begin{bmatrix}
\cos \pi x & -i \sin \pi x \\
-i \sin \pi x & \cos \pi x \\
\end{bmatrix}
\]

\[x(0) = c_1 + c_2 \]
\[x(L) = c_1 e^{i \pi x} + c_2 e^{-i \pi x} \]

Recall

\[x(x) = c_1 e^{i \pi x} + c_2 e^{-i \pi x} \]
Possibilities (1) & (2) are not meaningful.

So \( X(x) = 0 \) so \( u(x, t) = 0 \).

Solve \( c_1 = c_2 = 0 \) \( \Rightarrow \) Lin Syst has simple (steady state) ... 

So \( T(t) \) const.

So \( x^2 e^{x^2} \) \( \Rightarrow \) \( X(x) = c_1 + c_2 \).

\( \therefore \) Possibilities:

\( 1 \) \( \Rightarrow \) \( X(x) : 0 = k \)
3) $\lambda \geq 0$ can rewrite $X(x)$ in form

$$X(x) = d_1 \cos(\sqrt{\lambda} x) + d_2 \sin(\sqrt{\lambda} x)$$

**Dirichlet BVP:** $X(0) = d_1 = 0$

so $X(L) = d_2 \sin(\sqrt{\lambda} L) = 0$

Either: $d_2 = 0$ so $X(x) = 0$ so $u(x,t) = 0$ uninteresting

or: $\sqrt{\lambda} = \frac{n\pi}{L}$
... \sin \left( \frac{1}{n} x \right), \ n = 1, 2, 3, \ldots

\text{Conclusion:}

basics of solving for \( L \times L \)
$\cdots \quad \sin \left( \frac{\pi n x}{L} \right), \quad n = 0, 1, 2, \ldots$

$e^{-\beta (\frac{\pi n}{L})^2 t}$

\[ \mathcal{L} f(x) = \frac{\partial^2 f(x)}{\partial x^2} \]

But

\[ \text{Neumann BVP} \]

Basis of solutions for heat eqn

\[ \cdots \quad \sin \left( \frac{\pi n x}{L} \right), \quad n = 0, 1, 2, \ldots \]

\[ \text{basis of solutions for } X(x) \]

\[ \text{Similarly for Neumann BVP} \]
will solve IVP

\[ n = 1 \]

\[ \int_{0}^{\infty} P\left( \frac{\pi}{2n} \right) \frac{\sin \left( \frac{n\pi x}{L} \right)}{\sin \left( \frac{n\pi}{L} \right)} \, dx = f(x) \]

\[ n = 1, 2, 3, \ldots \]

If we can write \( f(x) \) so

\[ n(x,t) = \sum_{n=1}^{\infty} \int_{0}^{\infty} P\left( \frac{\pi}{2n} \right) \frac{\sin \left( \frac{n\pi x}{L} \right)}{\sin \left( \frac{n\pi}{L} \right)} \, dx \]

For Dirichlet BVP, \( n(x,0) \) is in "span" of \( \sin \left( \frac{n\pi x}{L} \right) \), \( n(x,0) = f(x) \)
Solve heat eqn on rod of length $L = \pi$

**Dirichlet BVP:** $u(x, t) = 0 = u(L, t)$

**Initial Condition:** $u(0, t) = 0$

**Initial Value Problem (IVP):** $u(x, t) = \left\{ \begin{array}{ll}
\frac{x}{\frac{\pi}{2}} & \text{if } 0 \leq x \leq \frac{\pi}{2} \\
\frac{\pi - x}{\frac{\pi}{2}} & \text{if } \frac{\pi}{2} \leq x \leq \pi
\end{array} \right.$
Need \( f(x) \)

Series of Fourier Sin

\[
\frac{2}{\pi x} \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right)
\]

\( n(x',0) = \int_{0}^{L} n(x,t) \, dx \)

How? Fourier Series!

If remains to determine \( b_n \)

\[
\frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx
\]

\( \text{Soln} \)

\[
\frac{1}{L} \int_{0}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx = \frac{2}{L} \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right)
\]
\[ y(x,t) = \frac{\pi}{4} \left( e^{-t} \sin(x) - \frac{9}{4} e^{-2t} \sin(3x) \right) \]

\[ + \frac{25}{4} e^{-2t} \sin(5x) - \cdots \]

Final answer:

\[ b_n = \begin{cases} 0 & n \text{ even} \\ \frac{\pi n}{h} \left( \frac{(-1)^{n}}{(n-1)!} \right) & n \text{ odd} \end{cases} \]

\[ \cdots \]