Lecture 23 1st order lin. systs of ODE

Friday Quiz through § 9.1

"The greatest shortcoming of the human race is our inability to understand the exponential function"

Albert Bartlett

Motivation for study of 1st order lin systs of ODE

Tradeoff More eqns, fewer derivs!

Consider nth order lin ODE

$$y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y^{(1)} + a_0 y = f$$

Goal: transform into 1st order lin syst of ODE

Introduce new fns

$$x_1 = y$$
$$x_2 = y^{(1)}$$
$$x_3 = y^{(2)}$$
$$\vdots$$
$$x_n = y^{(n-1)}$$

organize into vect. of fns

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Take <u>single</u> derivative:

$$\underline{x}' = \begin{bmatrix} x_1' \\ \vdots \\ \vdots \\ x_{n-1}' \\ x_n' \end{bmatrix} = \begin{bmatrix} y^{(1)} \\ \vdots \\ \vdots \\ y^{(n-1)} \\ y^{(n)} \end{bmatrix} = \begin{bmatrix} x_2 \\ \vdots \\ \\ x_n \\ -a_{n-1}y^{(n-1)} \cdots - a_0 y + f \end{bmatrix}$$

$$= \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ -a_{n-1}x_n \cdots - a_0 x_1 + f \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & & & \\ 0 & 0 & \cdots & & 0 & 1 \\ -a_0 & -a_1 & \cdots & & & -a_{n-1} \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{bmatrix}$$

__Def__ A 1st order lin syst of ODE is in __normal__ for__m__ if written as

$$\underline{x}' = A\,\underline{x} + \underline{b}$$

n-vect. of fns

$n \times n$ matrix

(we will focus on case when entries = numbers)

__Homog__: when $\underline{b} = \underline{0}$

Ex $A = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ **Solve** $\underline{x}' = A\underline{x}$

Soln Since A diag. eqns are indep.

$x_i' = \lambda_i x_i$ **Soln** $x_i = c_i e^{\lambda_i t}$

Basis of solns to **lin** syst $\underline{x}_1 = \begin{bmatrix} e^{\lambda t} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \underline{x}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ e^{\lambda_n t} \end{bmatrix}$

Exer Consider $y'' - 2y' - 3y = 0$

1) Find basis of solns
2) Rewrite as 1st order lin syst and also basis of solns

Soln 1) $r^2 - 2r - 3 = 0$ roots $r_1 = 3, r_2 = -1$

basis of solns $y_1 = e^{3t}, y_2 = e^{-t}$

2) $x_1 = y$ $\quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x_2 = y'$

$$\underline{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ 2y' + 3y \end{bmatrix}$$

$$= \begin{bmatrix} x_2 \\ 2x_2 + 3x_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \underline{x}$$

(homog)
normal
form!

Finally rewrite solns to original eqn as solns to normal form eqn

Basis of solns to orig eqn

$$y_1 = e^{3t} \qquad y_2 = e^{-t}$$

Basis of solns to normal form eqn

$$\underline{x}_1 = \begin{bmatrix} y_1 \\ y_1' \end{bmatrix} = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} = e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\underline{x}_2 = \begin{bmatrix} y_2 \\ y_2' \end{bmatrix} = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

What is special about $\underline{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$?

e-vectors of A with e-values $r_1 = 3$, $r_2 = -1$!

In other words: if we take basis $\beta = \{\underline{v}_1, \underline{v}_2\}$ and form $P = \begin{bmatrix} | & | \\ \underline{v}_1 & \underline{v}_2 \\ | & | \end{bmatrix}$ then

$$\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = P^{-1} A P \quad !$$

<u>Thm</u> Suppose $n \times n$ matrix A has basis of e-vectors $\underline{v}_1, \ldots, \underline{v}_n$ with e-values $\lambda_1, \ldots, \lambda_n$

Then $\underline{x}' = A\underline{x}$ has basis of solns

$$\underline{x}_1 = e^{\lambda_1 t}\underline{v}_1, \quad \ldots, \quad \underline{x}_n = e^{\lambda_n t}\underline{v}_n$$

<u>Pf</u> Check $\underline{x}_i = e^{\lambda_i t}\underline{v}_i$ is a soln

$$\underline{x}_i' = \lambda_i e^{\lambda_i t}\underline{v}_i$$

$$A\underline{x}_i = A e^{\lambda_i t}\underline{v}_i = e^{\lambda_i t} A \underline{v}_i$$

$$= e^{\lambda_i t}\lambda_i \underline{v}_i$$

= !

So indeed $\underline{x}_i$ is soln!

Lin indep: Suppose $c_1 \underline{x}_1 + \dots + c_n \underline{x}_n = 0$

Eval at $t=0$: $c_1 \underline{x}_1(0) + \dots + c_n \underline{x}_n(0)$

$= c_1 \underline{v}_1 + \dots + c_n \underline{v}_n = 0$

But $\underline{v}_1, \dots, \underline{v}_n$ lin indep so $c_1 = \dots = c_n = 0$.

so $\underline{x}_1, \dots, \underline{x}_n$ also lin indep!

Span Suppose soln $\underline{z} \notin \operatorname{Span}\{\underline{x}_1, \ldots, \underline{x}_n\}$

Then $\underline{z}, \underline{x}_1, \ldots, \underline{x}_n$ are lin indep.

But eval at t=0: $\underline{z}^{(0)} \in \operatorname{Span}\{\underset{\underline{x}_1(0)}{\underline{v}_1}, \ldots, \underset{\underline{x}_n(0)}{\underline{v}_n}\}$

since $\underline{v}_1, \ldots, \underline{v}_n$ span

So $\underline{z}(0), \underline{x}_1(0) = \underline{v}_1, \ldots, \underline{x}_n(0) = \underline{v}_n$

are **not** lin indep

Wronskian Lemma gives contradiction!

It says $\underline{z}, \underline{x}_1, \ldots, \underline{x}_n$ lin indep solns $\Rightarrow \underline{z}(0), \underline{x}_1(0), \ldots, \underline{x}_n(0)$ lin indep $\blacksquare$

Exer Consider $y'' - 4y' + 4 = 0$

1) Find basis of solns

2) Rewrite in normal form and rewrite basis of solns

Soln 1) $r^2 - 4r + 4 = 0$ roots $r_1 = r_2 = 2$

basis of **solns** $y_1 = e^{2t}$, $y_2 = te^{2t}$

2) $x_1 = y$

$x_2 = y'$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\underline{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ 4y' - 4y \end{bmatrix}$$

$$= \begin{bmatrix} x_2 \\ -4x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} \underline{x}$$

Basis of soln s

$$\underline{x}_1 = \begin{bmatrix} y_1 \\ y_1' \end{bmatrix} = \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix} = e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

e-vector $\underline{v}_1$

e-value = 2

But no other lin indep e-vectors!

$\dim E_2 = 1.$

Nevertheless can still use as second basis vector

$$\underline{x}_2 = \begin{bmatrix} y_2 \\ y_2' \end{bmatrix} = \begin{bmatrix} te^{2t} \\ e^{2t} + 2te^{2t} \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} t \\ 1 + 2t \end{bmatrix}$$

Does rewriting in <u>normal</u> <u>form</u> help?

<u>Thm</u> Given $\underline{x}' = A\underline{x}$ normal form eqn

Basis of solns is given by cols of matrix

$$X = e^{At}!$$

Not factorial but Wow!

<u>What is e^{At}?</u>

$$e^{At} = I_n + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

<u>Pf</u> $\frac{d}{dt}\left(e^{At}\right) = \frac{d}{dt}\left(I_n + At + \frac{(At)^2}{2!} + \dots\right)$

$$= 0 + A + \frac{A^2 t}{1!} + \frac{A^3 t^2}{2!} + \dots$$

$$= A\left(I + At + \frac{(At)^2}{2!} + \dots\right)$$

$$= A\,e^{At}$$

so cols are solns!

and in fact a basis...

Exer Solve $\underline{x}' = A\underline{x}$ where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

e-values of A are 0 with mult 3

but $\dim \underline{E}_0 = 1$ ...

so <u>no basis</u> of <u>e-vectors</u>

<u>available</u>

But can use matrix exponential

$$X = e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \frac{1}{2} \begin{bmatrix} 0 & 0 & t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

all higher terms $= 0$!

We find

$$X = \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

Basis of solns: cols of X

$$\underline{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \underline{x}_2 = \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$$

$$\underline{x}_3 = \begin{bmatrix} t^2/2 \\ t \\ 1 \end{bmatrix}$$