Lecture 21

Nonhomogeneous 2nd Order Linear ODEs

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Friday: No classes = No quiz!

Next Week: Back to normal.

"Guests bring good luck with them." Kurdish Proverb

Warm up

1) Find general soln to

$$y'' - 4y' + 5y = 0$$

2) Find a particular solution (IVP)

$$y(0) = 3 \quad y'(0) = -4$$

Soln:

Auxiliory equation

$$r^2 - 4r + 5 = 0$$

$$r = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

$r = 2 \pm i$

Actually, $\{e^{(2+i)t}, e^{(2-i)t}\}$

still a basis of solutions.

But, these are <u>complex</u>-valued functions!

<u>Euler's formula</u>: $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{(2+i)t} = e^{2t}e^{it} = e^{2t}(\cos t + i\sin t)$$

$\hookrightarrow \{e^{2t}\cos t, e^{2t}\sin t\}$ basis of real solns

Conclusion :

real general solution is

$$y(t) = C_1 e^{2t}\cos t + C_2 e^{2t}\sin t$$
$$= e^{2t}\left(C_1 \cos t + C_2 \sin t\right)$$

2) IVP: $y(0) = 3 \quad y'(0) = -4$

$$y'(t) = 2e^{2t}\left(C_1 \cos(t) + C_2 \sin(t)\right)$$
$$+ e^{2t}\left(-C_1 \sin(t) + C_2 \cos(t)\right)$$

$$3 = y(0) = e^0\left(C_1 \cos(0) + C_2 \sin(0)\right) = C_1$$

$$-4 = y'(0) = 2(C_1 + 0) + (0 + C_2)$$

$3 = c_1, \quad -4 = 2c_1 + c_2$

$-4 = 6 + c_2 \quad c_2 = -10$

Particular Soln:

$$y(t) = e^{2t}(3\cos t - 10\sin t)$$

Summary of homogeneous case:

$$y'' + by' + cy = 0$$

1) Always a 2-dimensional space of solutions

$y'' + by' + cy = 0 \rightsquigarrow$ Auxiliary equation
$r^2 + br + c = 0$

1) $r_1 \neq r_2$ distinct real roots
$\{e^{r_1 t}, e^{r_2 t}\}$ basis

2) One repeated root r
$\{e^{rt}, te^{rt}\}$ basis

3) Complex conjugate roots $\alpha \pm i\beta$
$\{e^{(\alpha+i\beta)t}, e^{(\alpha-i\beta)t}\}$ - basis of $\mathbb{C}$ solns
$\{e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)\}$ - basis of $\mathbb{R}$ solns.

Moreover:

<u>Wronskian lemma</u>: For a basis of solutions $\{y_1, y_2\}$

$$\begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{bmatrix}$$ is always invertible.

that is,

$$\left[\begin{array}{cc|c} y_1(0) & y_2(0) & A \\ y_1'(0) & y_2'(0) & B \end{array}\right]$$ always consistent

↳ Can solve
$y(0) = A$
$y'(0) = B$

On to nonhomogeneous 2nd order linear constant coefficient equations.

$$\underline{y'' + by' + cy = f}$$

$T: \frac{d^2}{dt^2} + b\frac{d}{dt} + cI$ linear transformation

$T: V \to V$ where V is the vector space of infinitely differentiable functions.

<u>Homogeneous:</u> $T(y) = 0$

<u>Non-homogeneous:</u> $T(y) = f$

$T(y) = 0$ vs. $T(y) = f$

How to Solve??

How did we solve $A\underline{x} = \underline{b}$?

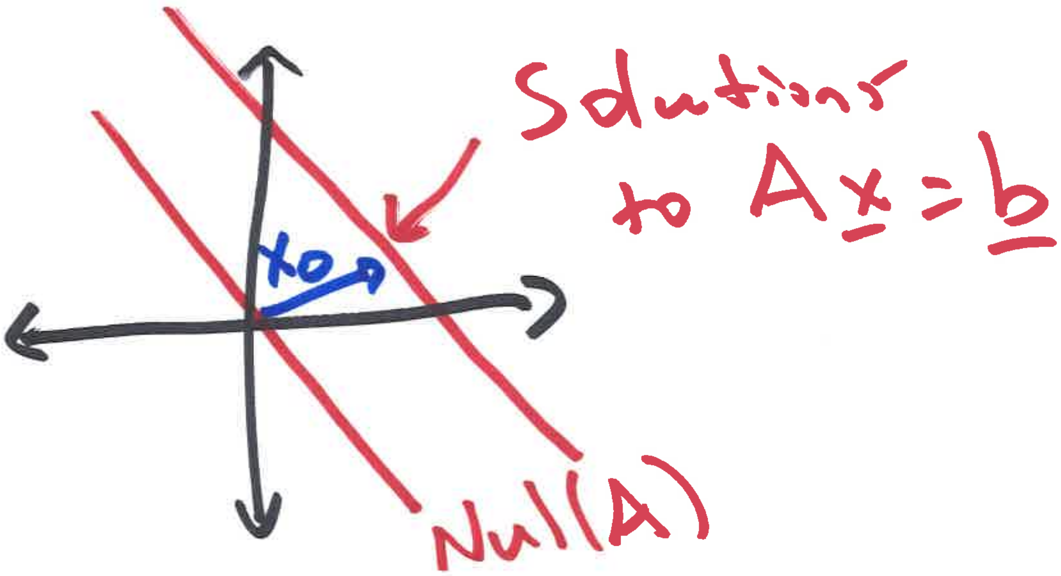
Row reduce? Not an option

Recall: $A\underline{x} = \underline{b}$

and $\underline{x}_0$ a particular soln (one soln)

We can always write $\underline{x} = \underline{x}_0 + \underline{v}$

where $\underline{v} \in \mathrm{Nul}(A)$.



to Solve $A\underline{x} = \underline{b}$

1) Find one soln $\underline{x}_0$

2) Solve $A\underline{x} = \underline{0}$

3) $\underline{x} = \underline{x}_0 + \underline{v}$ where $\underline{v}$ are solns from part 2.

When put $T = \frac{d^2}{dt^2} + b\frac{d}{dt} + cI$
Know how to do 2) and 3)

→ Figure out how to do 1)!

How are we going to find one solution to $y'' + by' + cy = f$?

<u>Answer</u>: Make a <u>good</u> <u>guess</u> Undetermined coefficients

We'll only consider <u>nice</u> functions:

Consider $f = t^n, e^{at}, \cos(at), \sin(at)$

Also, some products:

$t^n e^{at}, t^n \cos(at), t^n \sin(at)$.

Exer: Find a soln to $y'' + 3y' + 2y = t^2$

(**Note:** roots of auxiliary equation $-1, -2$)

$$T = \frac{d^2}{dt^2} + 3\frac{d}{dt} + 2I$$

$$\left.\begin{aligned} T: 1 &\longmapsto 2 \\ t &\longmapsto 3 + 2t \\ t^2 &\longmapsto 2 + 6t + 2t^2 \end{aligned}\right\}$$

t^2 is in the span of these polynomials!

Find $y(t) = A_2 t^2 + A_1 t + A_0$

Want to solve (Basis $\{t^2, t, 1\}$ of $\mathbb{P}_2$)

$$\begin{bmatrix} 2 & 0 & 0 \\ 6 & 2 & 0 \\ 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} A_2 \\ A_1 \\ A_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} | & | & | \\ T(t^2) & T(t) & T(1) \\ | & | & | \end{bmatrix} \begin{bmatrix} A_2 \\ A_1 \\ A_0 \end{bmatrix} = \left[t^2\right]$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 6 & 2 & 0 & 0 \\ 2 & 3 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & -3 \\ 0 & 3 & 2 & -1 \end{array}\right]$$

$A_2 = 1/2$

$A_1 = -3/2$

$3A_1 + 2A_0 = -1$

$2A_0 = -1 + 9/2 = 7/2 \quad A_0 = 7/4$

A particular soln to

$$y'' + 3y' + 2y = t^2$$

is $\underline{y(t) = \frac{1}{2}t^2 + \frac{-3}{2}t + 7/4}$.

~~for a so~~

↳ <u>General solution:</u>

$$y(t) = C_1 e^{-t} + C_2 e^{-2t} + \frac{1}{2}t^2 - \frac{3}{2}t + 7/4.$$

<u>Thm</u>: We can always solve (find a particular soln)

$$y'' + by' + cy = t^n \quad (c \neq 0)$$

with solution of the form

$$Y(t) = A_n t^n + \cdots + A_1 t + A_0$$

Find coefficients by solving

$$\begin{bmatrix} | & & | \\ T(t^n) & \cdots & T(1) \\ | & & | \end{bmatrix} \begin{bmatrix} A_n \\ \vdots \\ A_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

__Exer:__ Find a soln to

$$y'' + 3y' + 2y = e^{3t}$$

$$T: e^{3t} \longmapsto 9e^{3t} + 9e^{3t} + 2e^{3t}$$
$$= 20e^{3t}$$

A particular solution is

$$y(t) = \frac{e^{3t}}{20}\,!$$

Exer: Find a soln to

$$y'' + 3y' + 2y = \underline{\underline{e^{-2t}}}$$

← a solution to the homog. equation!

Guess e^{-2t}

$$T(e^{-2t}) = 0 \quad :($$

$$T(te^{-2t}) = \cancel{4te^{-2t}} - 4e^{-2t} + 3e^{-2t} \cancel{-6te^{-2t}} \cancel{+2te^{-2t}} = -e^{-2t}$$

$$\frac{d}{dt}(te^{-2t}) = e^{-2t} + (-2t)e^{-2t}$$

$$\frac{d^2}{dt^2}(te^{-2t}) = -2e^{-2t} - 2e^{-2t} + 4te^{-2t}$$

Solution is $y(t) = -te^{-2t}!$